Another Proof of Clairaut's Theorem

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Abstract. This note gives an alternate proof of Clairaut's theorem—that the partial derivatives of a smooth function commute—using the Stone–Weierstrass theorem.

Most calculus students have probably encountered Clairaut's theorem.

Theorem. *Suppose that* $f : [a, b] \times [c, d] \to \mathbb{R}$ *has continuous second-order partial derivatives. Then* $f_{xy} = f_{yx}$ *on* $(a, b) \times (c, d)$.

The proof found in many calculus textbooks (e.g., [**2**, p. A46]) is a reasonably straightforward application of the mean value theorem. More sophisticated techniques—Fubini's theorem and Green's theorem—can each be used to give easy proofs (for instance, [**1**, p. 61], exercise 3-28). The proof here relies on the density of two-variable polynomials in $C([a, b] \times [c, d])$. More precisely, we use the following version of the Stone–Weierstrass theorem.

Theorem. *Let* $g \in C([a, b] \times [c, d])$. *There is a sequence* $p_n(x, y)$ *of two-variable polynomials such that* $p_n \to g$ *uniformly.*

Applying the theorem to the continuous function f_{xy} gives a sequence of polynomials p_n such that

$$|p_n(x, y) - f_{xy}(x, y)| < \epsilon(n) \quad \text{for all} \quad (x, y) \in [a, b] \times [b, c]$$

where $\lim_{n\to\infty} \epsilon(n) = 0$.

Therefore, for any rectangle $R = [x_1, x_2] \times [y_1, y_2] \subset [a, b] \times [c, d]$,

$$\left| \iint_R p_n \, dxdy - \iint_R f_{xy} \, dxdy \right| < \epsilon(n) A(R), \tag{1}$$

where $A(R) = (x_2 - x_1)(y_2 - y_1)$ is the area of the rectangle R. Observe that

$$\iint_R f_{xy} \, dxdy = \iint_R f_{yx} \, dydx,$$

since both are equal to $f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)$.

Since p_n is a polynomial, it is a trivial computation to verify that

$$\iint_R p_n \, dxdy = \iint_R p_n \, dydx$$

for each $n \in \mathbb{N}$ (this also follows from Fubini's theorem, but even without assuming Fubini's theorem, equality is straightforward since both integrals can be directly computed).

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Therefore, we also have

$$\left| \iint_R p_n \, dydx - \iint_R f_{yx} \, dydx \right| < \epsilon(n) A(R). \tag{2}$$

Taking a limit as $n \to \infty$, (2) becomes

$$\iint_R f_{xy} - f_{yx} \, dydx = 0. \tag{3}$$

Since $f_{yx} - f_{xy}$ is continuous and (3) is true for all rectangles R, $f_{yx} - f_{xy}$ is identically zero, that is, $f_{xy} = f_{yx}$.

As a side remark, the same approach proves the equality of iterated integrals in the Fubini theorem for continuous functions. To see this, given $f \in C([a, b] \times [c, d])$, take $p_n \to f$, so

$$\int_c^d \int_a^b p_n(s, t) \, dsdt \to \int_c^d \int_a^b f dsdt$$
$$\int_a^b \int_c^d p_n(s, t) \, dtds \to \int_a^b \int_c^d f dtds.$$

As above, the two integrals on the left are equal for all n, so by uniqueness of limits, the right hand sides are also equal.

REFERENCES

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