An Extremely Short Proof of the Hairy Ball Theorem

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Abstract. Using winding numbers, we give an extremely short proof that every continuous field of tangent vectors on S^2 must vanish somewhere.

Consider the unit two sphere $S^2 = \{\mathbf{p} \in \mathbb{R}^3 : |\mathbf{p}| = 1\}$ in $\mathbb{R}^3$. We say a function $\mathbf{v} : S^2 \to \mathbb{R}^3$ is a *vector field* on S^2 if $\langle \mathbf{v}(\mathbf{p}), \mathbf{p} \rangle = 0$ for each $\mathbf{p} \in S^2$ and call a vector field *continuous* if its component functions are continuous.

Theorem 1. *Suppose* $\mathbf{v}$ *is a continuous vector field on* S^2. *Then there is* $\mathbf{p} \in S^2$ *such that* $\mathbf{v}(\mathbf{p}) = 0$.

This classical theorem was originally proven by Poincaré and is sometimes called the "Hairy Ball theorem." Theorem 1 has many interesting proofs (see, for instance, [**2**] and the charming book [**1**]) and various generalizations; for more information, see the introduction of [**2**]. The distinguishing attribute of the present proof is its brevity and elegance: Each of the aforementioned proofs requires computations in and between a set of stereographic coordinate charts that appropriately cover S^2. The argument here is shorter and and simpler.

A *regular smooth curve* in the plane is a smooth map $S^1 \to \mathbb{R}^2$ whose derivative does not vanish anywhere. The *rotation number* of such a curve γ is $\frac{1}{2\pi}$ times the change that the oriented angle $\dot{\gamma}$ makes with some fixed reference direction (e.g., $\mathbf{e}_1 = (1, 0)$) as the curve is traversed; in other words, it is the winding number of $\dot{\gamma}$, thought of as a map $S^1 \to \mathbb{R}^2 \setminus \{0\}$. The rotation number is an integer that is an invariant under *regular homotopy* (homotopy through regular curves).

Proof. Suppose for the sake of a contradiction that S^2 admits a continuous nonvanishing vector field $\mathbf{v}$; we may suppose $\mathbf{v}$ has unit length by replacing $\mathbf{v}$ with $\frac{\mathbf{v}}{|\mathbf{v}|}$. We first note that the definition of rotation number can be extended to curves in S^2 by replacing the *fixed* reference direction $\mathbf{e}_1$ by the *variable* direction $\mathbf{v}$ in the definition above.

To see this, endow $\mathbb{R}^3$ with a right-handed orientation so the ordered 3-tuple of standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is positively oriented and identify $\mathbb{R}^2$ with the subset $\{(x, y, z) \in \mathbb{R}^3 : z = 0\} \subset \mathbb{R}^3$. Given $\mathbf{p} \in S^2$ and a unit vector $\mathbf{w} \in T_{\mathbf{p}}S^2$, there is a unique unit vector $\mathbf{w}^\perp \in T_{\mathbf{p}}S^2$ such that $\{\mathbf{p}, \mathbf{w}, \mathbf{w}^\perp\}$ is positively oriented. For such $\mathbf{p}$ and $\mathbf{w}$, denote by $\Phi_{\mathbf{p},\mathbf{w}}$ the isometry of $\mathbb{R}^3$ determined by requesting that $\Phi_{\mathbf{p},\mathbf{w}}$ map the point $\mathbf{p}$ to $\mathbf{0}$ and send the ordered 3-tuple of tangent vectors $\{\mathbf{w}, \mathbf{w}^\perp, \mathbf{p}\} \subset T_{\mathbf{p}}\mathbb{R}^3$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset T_{\mathbf{0}}\mathbb{R}^3$. Clearly, $\Phi_{\mathbf{p},\mathbf{w}}$ depends continuously on $\mathbf{p}$ and $\mathbf{w}$. We define the rotation number of a curve γ in S^2 with respect to $\mathbf{v}$ to be the winding number of the continuous curve $\Phi_{\gamma,\mathbf{v}(\gamma)}(\dot{\gamma})$.

Consider now the family of regular smooth curves in S^2 defined as follows: $C_{\mathbf{p},s}$ (for $\mathbf{p} \in S^2$, $s \in (-1, 1)$) is the circle that is the intersection of S^2 and the plane $\{\mathbf{q} \in S^2 :$

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$\langle \mathbf{q}, \mathbf{p} \rangle = s\}$, oriented so that $\mathbf{p}$ is the positive normal. These curves are all regularly homotopic and so have the same rotation number with respect to $\mathbf{v}$, say n.

Now notice that for $s = 0$, $C_{\mathbf{p},s}$ and $C_{-\mathbf{p},s}$ parametrize the same great circle but with opposite orientations. Thus, $n = -n$ and hence $n = 0$. On the other hand, for s close to 1, the rotation number of $C_{\mathbf{p},s}$ is close to the rotation number of a circle in the plane because $\mathbf{v}$ is close to $\mathbf{v}(\mathbf{p})$ on $C_{\mathbf{p},s}$ by continuity. Thus, $n \in \{-1, 1\}$. This is a contradiction. ■

REFERENCES

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