AN ISOPERIMETRIC DEFICIT FORMULA

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ABSTRACT. We prove a geometric formula for the isoperimetric deficit of a smooth domain in a constant curvature model space.

If γ is a simple closed curve of length L in the plane bounding a region of area A, the classical isoperimetric inequality asserts that $L^2 - 4\pi A \geq 0$. More generally, if the curve γ lies in one of the constant curvature model spaces $\mathbb{H}^2, \mathbb{R}^2$, or $\mathbb{S}^2$, then $L^2 - 4\pi A + KA^2 \geq 0$, where K is the Gaussian curvature of the ambient space. In each case, equality holds precisely when γ is a circle.

A large body of work has focused on stability in the isoperimetric inequality. The basic idea of such work is to show that the isoperimetric deficit $L^2 - 4\pi A + KA^2$ bounds a nonnegative quantity measuring the asymmetry of the underlying curve γ, so that a curve with small isoperimetric deficit is then close to a circle in a quantitative way. In reference to Bonnesen's work [3] on the topic in the 1920s, Osserman [8] calls any inequality of the form $L^2 - 4\pi A + KA^2 \geq B$ a Bonnesen inequality, provided the quantity B is non-negative, has geometric significance, and vanishes only when γ is a geodesic circle.

Theorem. *Let $\Omega \subset M^2$ be a domain with C^1 boundary, where M^2 is $\mathbb{R}^2$, $\mathbb{S}^2$, or $\mathbb{H}^2$. Let A be the area of Ω and L the boundary length. Then*

$$L^2 - 4\pi A + KA^2 = \frac{1}{2} \int_{\partial\Omega \times \partial\Omega} |\nu_x - \mathsf{R}\nu_y|^2 \, ds_x ds_y, \tag{0.1}$$

where K is the curvature of M, ν is the unit outward pointing normal field along $\partial\Omega$, and R is the reflection in M sending x to y.

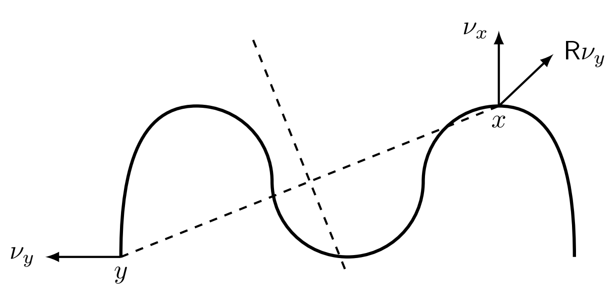

FIGURE 1. A portion of a boundary curve, two boundary points and their normals, and the reflected vector $\mathsf{R}\nu_y$.

On the right hand side of (0.1), we have defined a notion of asymmetry based on comparing the outward pointing normal vectors of the boundary points. To be more precise, note that given distinct points $x, y \in M^2$, where M^2 is either $\mathbb{H}^2, \mathbb{R}^2$, or $\mathbb{S}^2$, there is a unique involutive isometry $\mathsf{R} : M^2 \to M^2$ sending x to y. We call R a reflection. In the vicinity of a pair x, y of boundary points, $|\nu_x - \mathsf{R}\nu_y|^2$ measures the asymmetry of the domain under R.

Background.

When Ω is a convex domain in $\mathbb{R}^2$, (0.1) is equivalent to a formula first derived by Pleijel [9], who concludes after simplifying an integral over the space of oriented lines intersecting Ω that

$$L^2 - 4\pi A = 2 \int\limits_{\partial\Omega\times\partial\Omega} \sin^2 \tfrac{1}{2}(\theta_x - \theta_y)\, ds_x ds_y,$$

where θ_x and θ_y are the angles the tangents to $\partial\Omega$ make with the line passing through x and y. Using trigonometric identities, it is not difficult to see that (using the notation of (0.1)) $2\sin^2 \frac{1}{2}(\theta_x - \theta_y) = \frac{1}{2}|\nu_x - \mathsf{R}\nu_y|^2$.

Banchoff and Pohl [1] later generalized Pleijel's result, replacing $\partial\Omega$ with a general and possibly non-simple smooth curve γ in $\mathbb{R}^2$ and proving

$$L^2 - 4\pi \int_{\mathbb{R}^2} w^2(p)\, dA = 2 \int\limits_{\gamma\times\gamma} \sin^2 \tfrac{1}{2}(\theta_x - \theta_y)\, ds_x ds_y,$$

where given $p \in \mathbb{R}^2$, $w(p)$ is the winding number of γ about p. Other authors [7, 10] have proved isoperimetric inequalities in related settings by computing double boundary integrals.

Another notable result is due to Chakerian [4], who proves that the area A and boundary length L of a two-dimensional minimal surface in $\mathbb{R}^n$ bounded by a curve γ, normalized so that the position vector field X on $\mathbb{R}^n$ satisfies $\int_\gamma X ds = 0$, satisfy

$$L^2 - 4\pi A \geq \frac{2\pi^2}{L} \int_\gamma \left| X - \frac{L}{2\pi}\nu \right|^2 ds.$$

Beginning with the breakthrough work in [5], the last decade has seen a renewed interest in stability in the isoperimetric inequality. Fusco and Julin [6] have proved a stability inequality which is particularly interesting in comparison to (0.1). Although their results hold in much greater generality, in the special case that $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary and is normalized to have measure $|\Omega| = \pi$, Fusco and Julin define an *asymmetry index* $\mathcal{A}(\Omega)$ by

$$\mathcal{A}(\Omega) = \min_{y\in\mathbb{R}^2} \left\{ |\Omega \Delta B_1(y)| + \left(\int_{\partial\Omega} |\nu_\Omega(x) - \nu_{B_1(y)}(\pi_y(x))|^2 ds_x \right)^{1/2} \right\},$$

where $\Omega\Delta B_1(y)$ denotes the symmetric difference $(\Omega \setminus B_1(y)) \cup (B_1(y) \setminus \Omega)$, $B_1(y)$ is the unit ball centered at y, $\pi_y : \mathbb{R}^2 \setminus \{y\} \to B_1(y)$ is the radial projection, and prove that

$$\mathcal{A}(\Omega)^2 \leq C(L - 2\pi),$$

where C is a constant which is independent of Ω. We note that Bögelein, Duzaar, and Fusco [2] have recently proved an analogous inequality for domains in the sphere $\mathbb{S}^n$.

The Proof.

We first consider the euclidean case. It is convenient to define $r := |x - y|$ and denote derivatives with respect to x or y by appropriate subscripts, so that $\nabla_x r = -\nabla_y r = \frac{1}{r}(x - y)$. In this notation, we have that $\mathsf{R}\nu_y = \nu_y + 2\frac{\partial r}{\partial \nu_y}\nabla_x r$, where $\frac{\partial r}{\partial \nu_y} := \langle \nabla_y r, \nu_y \rangle$, so that

$$\frac{1}{2}\left|\nu_x - \mathsf{R}\nu_y\right|^2 = 1 - \left\langle \nu_y + 2\frac{\partial r}{\partial \nu_y}\nabla_x r, \nu_x \right\rangle.$$

Next, using that ν_y is a constant vector field with respect to x, compute

$$\begin{aligned}
\operatorname{div}_x\left(\frac{\partial r}{\partial \nu_y}\nabla_x r\right) &= \left\langle \nabla_x \frac{\partial r}{\partial \nu_y}, \nabla_x r \right\rangle + \frac{\partial r}{\partial \nu_y}\Delta_x r \\
&= (\nabla_x^2 r)(\nabla_x r, \nu_y) + \frac{\partial r}{\partial \nu_y}\Delta_x r \\
&= \frac{1}{r}\frac{\partial r}{\partial \nu_y},
\end{aligned}$$

where we have used that the hessian $\nabla_x^2 r$ satisfies $\nabla_x^2 r = r d\theta \otimes d\theta$ in local polar coordinates centered about x.

Combining the above items with the divergence theorem, we find

$$\frac{1}{2}\int_{x\in\partial\Omega}\left|\nu_x - \mathsf{R}\nu_y\right|^2 = L - 2\int_{x\in\Omega}\frac{1}{r}\frac{\partial r}{\partial \nu_y}.$$

Integrating again and using Fubini's theorem, we conclude that

$$\begin{aligned}
\frac{1}{2}\int_{\partial\Omega\times\partial\Omega}|\nu_x - \mathsf{R}\nu_y|^2\, ds_x ds_y &= L^2 - 2\int_{x\in\Omega}\int_{y\in\partial\Omega}\frac{1}{r}\frac{\partial r}{\partial \nu_y} \\
&= L^2 - 2\int_{x\in\Omega}\int_{y\in\Omega}\Delta_y \log r \\
&= L^2 - 4\pi A,
\end{aligned}$$

where we have used that $\frac{1}{r}\frac{\partial r}{\partial \nu_y} = \frac{\partial \log r}{\partial \nu_y}$ and that $\Delta_y \log r = 2\pi\delta_{x-y}$. This completes the proof of (0.1) in the euclidean setting.

We now consider the case where M^2 is either $\mathbb{S}^2$ or $\mathbb{H}^2$. For uniformity of presentation, we find the following notation convenient. We embed M^2 in $\mathbb{R}^3$ by

$$M^2 = \{x \in \mathbb{R}^3 : \langle x, x \rangle = 1\},$$

where in the case of $\mathbb{H}^2$, $\langle\,,\rangle$ denotes the Lorentzian scalar product on $\mathbb{R}^3 = \mathbb{R}^{2,1}$ defined by $\langle x, y\rangle := -x_1y_1 - x_2y_2 + x_3y_3$, and in the case of $\mathbb{S}^2$, $\langle\,,\rangle$ represents the usual euclidean inner product on $\mathbb{R}^3$. To avoid confusion, we denote the induced Riemannian metric on $\mathbb{H}^2$ with a dot $\cdot$, so that given $u, v \in T_x\mathbb{H}^2$, we have $u \cdot v := -\langle u, v\rangle$.

Notation 0.2. When we write an equation involving one or several instances of the expressions $\pm$ or $\mp$, we mean that the equation with the top argument(s) holds on $\mathbb{S}^2$ and that the equation with the bottom argument(s) holds on $\mathbb{H}^2$. For example, the isoperimetric inequality can be written $L^2 \geq 4\pi A \mp A^2$ in this notation.

The following lemma collects some facts used in the proof below.

Lemma 0.3. (i) $\frac{1}{2}|\nu_x - \mathsf{R}\nu_y|^2 = 1 \mp \langle \nu_y, \nu_x\rangle \mp \frac{\langle x, \nu_y\rangle\langle y, \nu_x\rangle}{1 - \langle x, y\rangle}$.

(ii) $\operatorname{div}_x \left(\nu_y^\top + \frac{\langle x, \nu_y\rangle}{1 - \langle x, y\rangle} y^\top \right) = -\frac{\langle x, \nu_y\rangle}{1 - \langle x, y\rangle}$.

(iii) $\operatorname{div}_y \left(\frac{x^\top}{1 - \langle x, y\rangle} \right) = 1 \mp 4\pi\delta$.

Proof. Since R is induced by the ambient Euclidean or Lorentzian reflection interchanging x and y,

$$\begin{aligned} \mathsf{R}\nu_y &= \nu_y - 2\langle \nu_y, x - y\rangle \frac{x - y}{|x - y|^2} \\ &= \nu_y - \frac{\langle \nu_y, x\rangle}{1 - \langle x, y\rangle}(x - y). \end{aligned}$$

Because $\mathsf{R}\nu_y$ and ν_x are unit vectors we then find (recall 0.2)

$$\begin{aligned} \frac{1}{2}|\nu_x - \mathsf{R}\nu_y|^2 &= 1 \mp \langle \mathsf{R}\nu_y, \nu_x\rangle \\ &= 1 \mp \left\langle \nu_y - \frac{\langle x, \nu_y\rangle}{1 - \langle x, y\rangle}(x - y), \nu_x \right\rangle \\ &= 1 \mp \left\langle \nu_y + \frac{\langle x, \nu_y\rangle}{1 - \langle x, y\rangle} y, \nu_x \right\rangle. \end{aligned}$$

The following observation will be useful below: given a constant vector field w on $\mathbb{R}^3$, its tangential projection $w^\top := w - \langle w, x\rangle x$ to M^2 satisfies

$$\operatorname{div}_x(w^\top) = -2\langle x, w\rangle. \tag{0.4}$$

This is easily checked in local coordinates; when $M = \mathbb{S}^2$, (0.4) is an immediate consequence of the fact that the coordinate functions x^i satisfy $(\Delta_{\mathbb{S}^2} + 2)x^i = 0$.

For (ii) we compute using the product rule and (0.4)

$$\begin{aligned}
\operatorname{div}_x\left(\nu_y^\top + \frac{\langle x,\nu_y\rangle y^\top}{1-\langle x,y\rangle}\right) &= -2\langle \nu_y, x\rangle + \frac{\langle \nu_y^\top, y^\top\rangle - 2\langle x,\nu_y\rangle\langle x,y\rangle}{1-\langle x,y\rangle} + \frac{\langle x,\nu_y\rangle |y^\top|^2}{(1-\langle x,y\rangle)^2}\\
&= -2\langle \nu_y, x\rangle + \frac{\langle \nu_y^\top, y^\top\rangle - 2\langle x,\nu_y\rangle\langle x,y\rangle + \langle x,\nu_y\rangle(1+\langle x,y\rangle)}{1-\langle x,y\rangle}\\
&= -2\langle \nu_y, x\rangle + \frac{\langle \nu_y^\top, y^\top\rangle - \langle x,\nu_y\rangle\langle x,y\rangle + \langle x,\nu_y\rangle}{1-\langle x,y\rangle}\\
&= -\langle \nu_y, x\rangle + \frac{\langle \nu_y^\top, y^\top\rangle}{1-\langle x,y\rangle} = -\frac{\langle x,\nu_y\rangle}{1-\langle x,y\rangle},
\end{aligned}$$

where we have used that $|y^\top|^2 = 1-\langle x,y\rangle^2$ and $\langle \nu_y^\top, y^\top\rangle = -\langle x,y\rangle\langle x,\nu_y\rangle$ to simplify.

For (iii), compute also using (0.4)

$$\operatorname{div}_y\left(\frac{x^\top}{1-\langle x,y\rangle}\right) = \frac{-2\langle x,y\rangle}{1-\langle x,y\rangle} + \frac{\langle x^\top, x^\top\rangle}{(1-\langle x,y\rangle)^2} = 1,$$

where we have used that $\langle x^\top, x^\top\rangle = 1-\langle x,y\rangle^2 = (1-\langle x,y\rangle)(1+\langle x,y\rangle)$. This completes the calculation in (iii) for $x \neq y$.

Now fix $x \in M^2$ and let $U \subset M^2$ be a domain containing x. By the divergence theorem,

$$\int_{\partial U} \frac{x^\top \cdot \nu_y}{1-\langle x,y\rangle} = \int_{U\setminus B_\varepsilon(x)} \operatorname{div}_y\left(\frac{x^\top}{1-\langle x,y\rangle}\right) + \int_{\partial B_\varepsilon(x)} \frac{x^\top\cdot\nu_y}{1-\langle x,y\rangle},$$

where the last ν_y is the outward pointing normal to the $B_\varepsilon(x)$. We now consider the cases $M=\mathbb{S}^2$ and $M=\mathbb{H}^2$ separately. When $M=\mathbb{S}^2$, elementary geometry shows that on $\partial B_\varepsilon(x)$, $\nu_y = (\langle x,y\rangle y - x)/(1-\langle x,y\rangle^2)^{1/2}$. Using this and that $x^\top\cdot\nu_y = \langle x,\nu_y\rangle$ on $\mathbb{S}^2$, we find

$$\begin{aligned}
\lim_{\varepsilon\searrow 0}\int_{\partial B_\varepsilon(x)} \frac{x^\top\cdot\nu_y}{1-\langle x,y\rangle} &= \lim_{\varepsilon\searrow 0}\int_{\partial B_\varepsilon(x)} \frac{\langle x,y\rangle^2-1}{(1-\langle x,y\rangle)(1-\langle x,y\rangle^2)^{1/2}}\\
&= -\lim_{\varepsilon\searrow 0}\int_{\partial B_\varepsilon(x)} \left(\frac{1+\langle x,y\rangle}{1-\langle x,y\rangle}\right)^{1/2}\\
&= -4\pi,
\end{aligned}$$

where we have used that $\langle x,y\rangle = \cos\varepsilon$ on $\partial B_\varepsilon(x)$. A similar calculation shows the required result on $M=\mathbb{H}^2$, using there that on $\partial B_\varepsilon(x)$, $\langle x,y\rangle = \cosh\varepsilon$ and $\nu_y = (\langle x,y\rangle y - x)/(\langle x,y\rangle^2-1)^{1/2}$. $\square$

We are now ready to complete the proof of the theorem in the nonzero curvature case. Using Lemma 0.3 we have

$$\begin{aligned}\frac{1}{2}\int_{x\in\partial\Omega} |\nu_x - \mathsf{R}\nu_y|^2 &= L \mp \int_{x\in\partial\Omega} \left\langle \nu_y + \frac{\langle x,\nu_y\rangle}{1-\langle x,y\rangle} y, \nu_x \right\rangle \\ &= L - \int_{x\in\Omega} \operatorname{div}_x \left(\nu_y^\top + \frac{\langle x,\nu_y\rangle}{1-\langle x,y\rangle} y^\top \right) \\ &= L - \int_{x\in\Omega} \frac{\langle x,\nu_y\rangle}{1-\langle x,y\rangle}.\end{aligned}$$

Integrating over $y \in \partial\Omega$ and changing the order of integration, we have then

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega\times\partial\Omega} |\nu_x - \mathsf{R}\nu_y|^2 \, ds_x ds_y &= L^2 - \int_{x\in\Omega}\int_{y\in\partial\Omega} \frac{\langle x,\nu_y\rangle}{1-\langle x,y\rangle} \\ &= L^2 \mp \int_{x\in\Omega}\int_{y\in\Omega} \operatorname{div}_y \left(\frac{x^\top}{1-\langle x,y\rangle} \right) \\ &= L^2 \pm \int_{x\in\Omega}\int_{y\in\Omega} 1 \mp 4\pi\delta \\ &= L^2 \pm A^2 - 4\pi A.\end{aligned}$$

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