On the Smooth Jordan Brouwer Separation Theorem

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Abstract. We give an elementary proof of the Jordan Brouwer separation theorem for smooth hypersurfaces using the divergence theorem and the inverse function theorem.

Theorem 1. *Let $\Sigma \subset \mathbb{R}^n$ be a smooth, closed, connected, orientable hypersurface. $\mathbb{R}^n - \Sigma$ is a disjoint union of two connected open sets each of which has boundary Σ.*

The standard differential topology proof of Theorem 1 (e.g., see [**2**]) uses mod 2 intersection theory and requires Sard's theorem and the transversality theorems as necessary technical ingredients. Here, we forgo Sard's theorem and instead use the fundamental solution of Laplace's equation and the divergence theorem for technical power. Our proof uses the tubular neighborhood theorem to establish local candidate "inside" and "outside" sets that we are able to consistently extend to all of $\mathbb{R}^n - \Sigma$ by the divergence theorem.

We call a subset $\Sigma \subset \mathbb{R}^n$ a *smooth hypersurface* if for each $p \in \Sigma$ there is an open set $U \subset \mathbb{R}^n$ containing p and a smooth map $\Psi : U \to \mathbb{R}$ such that $\nabla\Psi(p) \neq 0$ and $\Psi^{-1}(0) = \Sigma \cap U$. We say a vector $v \in \mathbb{R}^n$ is *normal* to Σ at p if v is parallel to $\nabla\Psi(p)$. Finally, we say Σ is *orientable* if there exists a smooth vector field $\vec{n} : \Sigma \to \mathbb{R}^n$ such that $|\vec{n}_p| = 1$ and $\vec{n}_p$ is normal to Σ for each $p \in \Sigma$.

Proof. Fix a choice of a unit normal vector field $\vec{n}$ on Σ. A corollary of the inverse function theorem and compactness of Σ is the tubular neighborhood theorem.

Lemma 1. There exists $\epsilon > 0$ such that $\Sigma \times (-\epsilon, \epsilon)$ is diffeomorphic to

$$N_\epsilon = \{p + t\vec{n}_p : |t| < \epsilon,\ p \in \Sigma\} \subset \mathbb{R}^n$$

by the map $(p, t) \mapsto p + t\vec{n}_p$. Moreover, each point $q = p + t\vec{n}_p$ in N_ϵ has a unique closest point in Σ, namely p.

The proof is simple; the interested reader may find it for instance in [**3**].

For a subset $S \subset (-\epsilon, \epsilon)$, we denote Σ_S by the image of $\Sigma \times S$ under the above diffeomorphism; likewise, we denote Σ_s by the image of $\Sigma \times \{s\}$ for $|s| < \epsilon$.

It will be convenient to use the fundamental solution of Laplace's equation $\Phi : \mathbb{R}^n/\{0\} \to \mathbb{R}$ defined by

$$\Phi(y) = \begin{cases} \dfrac{1}{2\pi} \log|y| & (n = 2) \\ \dfrac{1}{(2-n)\omega_n} |y|^{2-n} & (n > 2). \end{cases}$$

http://dx.doi.org/10.4169/amer.math.monthly.123.3.292
MSC: Primary 58A05, Secondary 57R99

We use below the standard fact (cf. [**1**], p. 22) that for each fixed $x \in \mathbb{R}^n$, the function $\Phi_x(y) := \Phi(y - x)$ satisfies $\Delta\Phi_x = 0$, where the laplacian is taken with respect to y.

We fix some notation. For an open bounded subset $U \subset \mathbb{R}^n$ for which ∂U is C^1, we denote by ν the unit outward normal vector field on ∂U. For smooth functions f, we denote $\frac{\partial f}{\partial \nu} := \nabla f \cdot \nu$. For a vector field $\mathbf{F}$ on U, we recall the divergence theorem:

$$\int_U \operatorname{div}(\mathbf{F}) = \int_{\partial U} F \cdot \nu.$$

Of course, when $U \subset \mathbb{R}^3$, this is the familiar version from multivariable calculus (e.g., [**4**], section 16.9). Taking $\mathbf{F} = \nabla f$, the divergence theorem reads $\int_U \Delta f = \int_{\partial U} \frac{\partial f}{\partial \nu}$. Also, for $v \in \mathbb{R}^n$, denote by $T_v(M) = \{p + v : p \in M\}$ the translation of M by the vector v. We endow the surfaces Σ_s for $|s| < \epsilon$ and their translates the natural choice of a unit normal coming from $\vec{n}$ and denote it by the same symbol by abuse of notation.

Define $F_\Sigma : \mathbb{R}^n - \Sigma \to \mathbb{R}$ by

$$F_\Sigma(x) = \int_\Sigma \frac{\partial \Phi_x}{\partial \vec{n}}.$$

Lemma 2. F_Σ is locally constant.

Proof. Fix $x \notin \Sigma$ and an ϵ_1 with $0 < \epsilon_1 < \epsilon$ such that $x \notin \overline{\Sigma_{(0,\epsilon_1)}}$. Then Σ_{ϵ_1} and Σ bound the open set $\Sigma_{(0,\epsilon_1)}$. By the divergence theorem and the fact that $\Delta\Phi_x = 0$ on $\Sigma_{(0,\epsilon_1)}$, it follows that

$$\int_{\Sigma_{\epsilon_1}} \frac{\partial \Phi_x}{\partial \vec{n}} = \int_\Sigma \frac{\partial \Phi_x}{\partial \vec{n}}. \tag{1}$$

Now let v be a vector in $\mathbb{R}^n$ with $|v| < \epsilon_1$. Then $T_v(\Sigma)$ and Σ_{ϵ_1} bound an open subset of $\Sigma_{(-\epsilon_1,\epsilon_1)}$, so using the divergence theorem in the same way yields

$$\int_{\Sigma_{\epsilon_1}} \frac{\partial \Phi_x}{\partial \vec{n}} = \int_{T_v(\Sigma)} \frac{\partial \Phi_x}{\partial \vec{n}}. \tag{2}$$

Combining (1) and (2) gives us

$$\int_\Sigma \frac{\partial \Phi_x}{\partial \vec{n}} = \int_{T_v(\Sigma)} \frac{\partial \Phi_x}{\partial \vec{n}}, \tag{3}$$

which after the change of variables $y \mapsto y + v$ gives $F_\Sigma(x) = F_\Sigma(x - v)$. ■

Define

$$\Omega_1 = \{x \in \mathbb{R}^n - \Sigma : x \text{ is path connected to } \infty\}.$$

Because Σ is compact, there is a sufficiently large $R > 0$ such that the ball $B_R(0)$ contains Σ. Hence, Ω_1 is well defined, open, and connected. By shrinking R, we may assume that $\partial B_R(0) \cap (N_\epsilon \setminus \Sigma) \neq \emptyset$. Since the sets $\Sigma_{(0,\epsilon)}$ and $\Sigma_{(-\epsilon,0)}$ are path connected, it is easy to see Ω_1 contains at least one of the sets $\Sigma_{(0,\epsilon)}$ and $\Sigma_{(-\epsilon,0)}$; after possibly changing the choice of $\vec{n}$, we suppose $\Sigma_{(0,\epsilon)} \subset \Omega_1$.

Lemma 3. $F_\Sigma(x) = 0$ for $x \in \Omega_1$.

Proof. Compute $\nabla\Phi_x = \frac{1}{\omega_n|y-x|^n}(y - x)$ so for large $|x|$,

$$F_\Sigma(x) = \int_\Sigma \frac{1}{\omega_n|y-x|^n}(y-x)\cdot\vec{n}.$$

Then by the Cauchy–Schwarz inequality,

$$|F_\Sigma(x)| \le \int_\Sigma \frac{1}{\omega_n|y-x|^{n-1}}$$

and so $\lim_{|x|\to\infty} F_\Sigma(x) = 0$. The lemma follows by using Lemma 2 and taking $|x| \to \infty$. ■

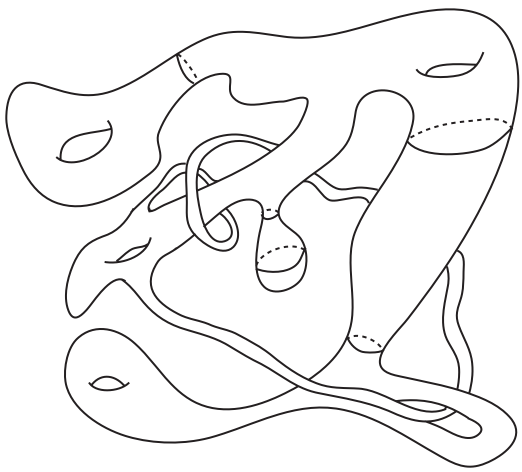

Figure 1. A cartoon representation of an "exotic" surface $\Sigma \subset \mathbb{R}^3$ to which Theorem 1 applies.

Now fix $x \in \Sigma_{(-\epsilon_1,0)}$ and pick $\delta > 0$ small enough so that $B(x,\delta) \subset \Sigma_{(-\epsilon_1,0)}$. Applying the divergence theorem to the open set bounded by $\partial B(x,\delta)$, Σ and $\Sigma_{-\epsilon_1}$ we have

$$\int_\Sigma \frac{\partial\Phi_x}{\partial\vec{n}} + \int_{\Sigma_{-\epsilon_1}} \frac{\partial\Phi_x}{\partial\nu} + \int_{\partial B(x,\delta)} \frac{\partial\Phi_x}{\partial\nu} = 0.$$

By the argument in Lemma 3 with $\Sigma_{-\epsilon_1}$ taking the role of Σ, $\int_{\Sigma_{-\epsilon_1}} \frac{\partial\Phi_x}{\partial\nu} = 0$. A direct computation shows $\int_{\partial B(x,\delta)} \frac{\partial\Phi_x}{\partial\nu} = 1$ so then $F_\Sigma(x) = -1$.

Define

$$\Omega_2 = \{x \in \mathbb{R}^n - \Sigma : F_\Sigma(x) = -1\}.$$

By the above calculation, $\Sigma_{(-\epsilon,0)} \subset \Omega_2$. Combining this with the fact $\Sigma_{(0,\epsilon)} \subset \Omega_1$ and the observation that $\mathbb{R}^n - \Sigma$ splits into path components, each of which has boundary contained in Σ, it follows that $\mathbb{R}^n - \Sigma = \Omega_1 \cup \Omega_2$ and the theorem is proved. ■

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An Identity of Carlitz and Its Generalization

In [**1**], Carlitz asked the readers to show that

$$\sum_{k=0}^{n-1} F_k 2^{n-k-1} = 2^n - F_{n+2} \quad \text{and} \quad \sum_{k=0}^{n-1} L_k 2^{n-k-1} = 3(2^n) - L_{n+2},$$

where F_n and L_n are the nth Fibonacci and Lucas numbers.

In this note we generalize these results to the k-Horadam sequence $H_{k,i}$, which is defined as follows [**2**]: for $n \geq 0$, if k is any positive real number and $f(k)$, $g(k)$ are scaler-value polynomials, with $f^2(k) + 4g(k) > 0$, then $H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$, with $H_{k,0} = a,\ H_{k,1} = b$, and $a, b \in \mathbb{R}$.

Let $\mathcal{S} = \sum_{i=0}^{n-1} H_{k,i} x^{n-i-1}$, then

$$\mathcal{S} = \frac{ax^{n+1} + (b - af(k))x^n - H_{k,n}x - g(k)H_{k,n-1}}{x^2 - f(k)x - g(k)}. \tag{4}$$

In fact,

$$\begin{aligned}
\mathcal{S} &= ax^{n-1} + bx^{n-2} + \sum_{i=2}^{n-1}(f(k)H_{k,i-1} + g(k)H_{k,i-2})x^{n-i-1} \\
&= ax^{n-1} + bx^{n-2} + f(k)x^{-1}\sum_{i=1}^{n-2} H_{k,i}x^{n-i-1} + g(k)x^{-2}\sum_{i=0}^{n-3} H_{k,i}x^{n-i-1} \\
&= ax^{n-1} + bx^{n-2} + f(k)x^{-1}(\mathcal{S} - H_{k,0}x^{n-1} - H_{k,n-1}) \\
&\quad + g(k)x^{-2}(\mathcal{S} - H_{k,n-2}x - H_{k,n-1}).
\end{aligned}$$

Multiplying both sides of the above equation by x^2 and solving for $\mathcal{S}$, we then obtain Equation (4).

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http://dx.doi.org/10.4169/amer.math.monthly.123.3.295
MSC: Primary 11B39, Secondary 11B83