On Liouville's Theorem for Conformal Maps

Mathew Kushelman and Peter McGrath

Abstract. A theorem of Liouville asserts that the simplest angle-preserving transformations on Euclidean space—translations, dilations, reflections, and inversions—generate all angle-preserving transformations when the dimension is at least 3. This note gives a proof which uses only elementary multivariable calculus and simplifies a differential-geometric argument of Flanders.

1. INTRODUCTION. The angle $\angle(v, w)$ between vectors $v, w \in \mathbb{R}^n$ is defined by

$$\angle(v, w) = \cos^{-1} \frac{v \cdot w}{|v||w|}.$$

A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **angle-preserving** if $\angle(Tv, Tw) = \angle(v, w)$ for all $v, w \in \mathbb{R}^n$, and a differentiable map $f : U \to \mathbb{R}^n$ defined on an open subset $U \subset \mathbb{R}^n$ is called **conformal** if its derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is angle-preserving for each $x \in U$.

It is convenient to express the angle-preserving condition in terms of an equivalent condition on lengths. This is done in the following lemma.

Lemma 1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The following are equivalent.*

1. *T is angle-preserving.*
2. *There exists $\alpha > 0$ such that $|Tv| = \alpha|v|$ for all $v \in \mathbb{R}^n$.*
3. *There exists $\beta > 0$ such that $Tv \cdot Tw = \beta v \cdot w$ for all $v, w \in \mathbb{R}^n$.*

Proof. If (1) holds, then the triangle formed by v, w and the origin is similar to the triangle formed by Tv, Tw, and the origin; in particular, all side lengths are scaled by some factor α, and (2) follows. If (2) holds, then for $v, w \in \mathbb{R}^n$,

$$Tv \cdot Tw = \frac{|T(v+w)|^2}{2} - \frac{|T(v-w)|^2}{2} = \alpha^2 \frac{|v+w|^2 - |v-w|^2}{2} = \alpha^2 v \cdot w,$$

so (3) holds. Finally, if (3) holds, then for $v, w \in \mathbb{R}^n$,

$$\cos\angle(Tv, Tw) = \frac{Tv \cdot Tw}{|Tv||Tw|} = \frac{\beta v \cdot w}{\sqrt{\beta}|v|\sqrt{\beta}|w|} = \cos\angle(v, w),$$

so (1) holds. ∎

By Lemma 1, associated to each conformal map is a function $\lambda : U \to \mathbb{R}$ satisfying

$$Dfv \cdot Dfw = e^{2\lambda} v \cdot w \quad \text{for all} \quad v, w \in \mathbb{R}^n.$$

The function e^λ is called the **conformal factor** associated to f.

The simplest conformal maps are the translations ($x \mapsto x + b$, $b \in \mathbb{R}^n$), dilations ($x \mapsto \alpha x, \alpha \neq 0$), and orthogonal transformations ($x \mapsto Tx$ for T a linear map with $|Tx| = |x| \ \forall x \in \mathbb{R}^n$). These maps generate a group of conformal transformations under function composition, called the group of **similarities**. Note that each similarity has constant conformal factor.

A less obvious conformal transformation is the **inversion** $i_a : \mathbb{R}^n \setminus \{a\} \to \mathbb{R}^n$ about the unit sphere centered at $a \in \mathbb{R}^n$ defined by

$$i_a(x) = a + \frac{x-a}{|x-a|^2}.$$

A straightforward calculation shows that the differential of i_a is given by

$$Di_a(x)v = \frac{R_{x-a}v}{|x-a|^2}, \quad \text{where} \quad R_{x-a}v := v - 2v \cdot \frac{x-a}{|x-a|}\frac{x-a}{|x-a|}.$$

Observe that R_{x-a} is the reflection through the hyperplane orthogonal to $x - a$, so

$$|Di_a(x)v| = \frac{|v|}{|x-a|^2} \quad \text{for all } v \in \mathbb{R}^n. \tag{1}$$

Thus, by Lemma 1, the linear map $Di_a(x)$ is angle-preserving for each x, hence i_a is conformal. Note that each inversion is an involution: $i_a = i_a^{-1}$.

The similarities and inversions generate a group of conformal transformations under function composition, called the **Möbius transformations**.

Remark 2. By (1) and the fact that the conformal factor for any similarity is constant, we see that the reciprocal ρ of the conformal factor of any Möbius transformation is of the form $\rho = \frac{1}{2}c|x-a|^2 + b$ for some $a \in \mathbb{R}^n$, $b, c \in \mathbb{R}$ and one of b, c equal to zero.

We now have what we need to state Liouville's theorem:

Theorem 3 (Liouville). *Let $f : U \to \mathbb{R}^n$ ($n \geq 3$) be a conformal transformation of class C^3, with $U \subset \mathbb{R}^n$ a connected open set. Then f is a composition of translations, dilations, orthogonal transformations, and inversions.*

2. THE PROOF. The proofs of Liouville's theorem found in most modern textbooks (for example [**1, 2, 3**]) are variations on an argument of Nevanlinna [**7**], the first step of which is to show that second partial derivatives ρ_{ij} of the reciprocal ρ of the conformal factor of any conformal map satisfy the equation

$$\rho_{ij} = c\delta_{ij} \tag{2}$$

for some constant c. This is motivated by Remark 2, which shows (2) certainly holds for any Möbius transformation. Integrating (2) shows that $\rho = \frac{1}{2}c|x-a|^2 + b$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and the second major step of Nevanlinna's proof is to show that either $b = 0$ or $c = 0$, so ρ is the conformal factor of a Möbius transformation. This is achieved by comparing lengths of line segments to lengths of their conformal images and appeals to the transcendentality of the logarithm and arctangent functions.

In 1966, Flanders [**4**] found a simpler proof which avoids the subtle argument just mentioned, which he described as a "rather out-of-context point in Nevanlinna's proof." Unfortunately, his differential-geometric proof, which uses exterior differential forms and structure and integrability conditions, has not become more popular.

The proof here simplifies and shortens Flanders' argument and importantly requires only elementary multivariable calculus. It begins along similar lines to Nevanlinna's, but in addition to (2) derives the equation $\rho(\rho_{ii} + \rho_{jj}) = |\nabla\rho|^2$, which eliminates the integration constant b if $c \neq 0$. This equation, as well as equation (2), follow from the commutivity of the third partial derivatives of the conformal map.

Proof of Liouville's theorem. Because f is conformal, we have

$$f_i \cdot f_j = e^{2\lambda}\delta_{ij} \quad 1 \le i, j \le n, \tag{3}$$

where we denote the partial derivatives of $f = (f^1, \dots, f^n)$ by appropriate subscripts, so $f_i = \partial f/\partial x_i = (\partial f^1/\partial x_i, \dots, \partial f^n/\partial x_i)$, $f_{ij} = f_{ji} = \partial^2 f/\partial x_i \partial x_j$. Considering three instances of (3) and taking partial derivatives reveals that

$$\begin{aligned} f_{ik} \cdot f_j + f_i \cdot f_{jk} &= 2e^{2\lambda}\lambda_k \delta_{ij}, \\ f_{ji} \cdot f_k + f_j \cdot f_{ki} &= 2e^{2\lambda}\lambda_i \delta_{jk}, \\ f_{kj} \cdot f_i + f_k \cdot f_{ij} &= 2e^{2\lambda}\lambda_j \delta_{ki} \end{aligned}$$

for any $1 \le k \le n$; adding the second and third equations and subtracting the first shows

$$f_{ij} \cdot f_k = e^{2\lambda}(\lambda_i \delta_{jk} + \lambda_j \delta_{ki} - \lambda_k \delta_{ij}).$$

Since the vectors f_k form an orthogonal basis at each point and $f_k \cdot f_k = e^{2\lambda}$,

$$\begin{aligned} f_{ij} &= \lambda_i f_j + \lambda_j f_i, \quad i \ne j \\ f_{ii} &= \lambda_i f_i - \sum_{k \ne i} \lambda_k f_k. \end{aligned} \tag{4}$$

Since f has continuous third-order partial derivatives, calculating using (4) reveals

$$\begin{aligned} 0 = f_{iji} - f_{iij} &= \lambda_{ii} f_j + \lambda_i f_{ji} + \lambda_{ji} f_i + \lambda_j f_{ii} \\ &\quad - \lambda_{ij} f_i - \lambda_i f_{ij} + \sum_{k \ne i} \lambda_{kj} f_k + \sum_{k \ne i} \lambda_k f_{kj} \\ &= (\lambda_{ii} + \lambda_{jj} + \sum_{k \ne i,j} \lambda_k^2) f_j + \sum_{k \ne i,j} (\lambda_{kj} - \lambda_j \lambda_k) f_k \end{aligned}$$

for each pair of distinct indices i, j. Since this holds for all such pairs,

$$\lambda_{ii} + \lambda_{jj} = -|\nabla \lambda|^2 + \lambda_i^2 + \lambda_j^2, \qquad \lambda_{ij} = \lambda_i \lambda_j, \quad i \ne j. \tag{5}$$

Now set $\rho = e^{-\lambda}$; we have

$$\lambda_i = -\frac{\rho_i}{\rho}, \quad \lambda_{ij} = -\frac{\rho_{ij}}{\rho} + \frac{\rho_i \rho_j}{\rho^2},$$

so (5) becomes

$$\rho(\rho_{ii} + \rho_{jj}) = |\nabla \rho|^2, \qquad \rho_{ij} = 0. \tag{6}$$

The first equation implies all the ρ_{ii} are equal, while the second implies ρ_i is a function of x_i alone; therefore

$$\rho_{ij} = c\delta_{ij}, \quad c \in \mathbb{R}.$$

After integrating this equation, we find for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ that

$$\rho = \frac{1}{2}c|x-a|^2 + b.$$

Case 1: $c = 0$. Then ρ and λ are constant, hence from (4) each f_i is constant, so f is an affine linear transformation. It is then easy to see f is a similarity.

Case 2: $c \neq 0$. Then the first equation in (6) implies $b = 0$; it follows that

$$|Df(x)v| = \frac{2}{c}\frac{|v|}{|x-a|^2} \quad \text{for all } v \in \mathbb{R}^n. \tag{7}$$

Consider now the inversion i_a. By the chain rule, (7), and (1),

$$|D(f \circ i_a)(x)v| = |Df(i_a(x)) \circ Di_a(x)v| = \frac{2}{c}\frac{|Di_a(x)v|}{|i_a(x)-a|^2} = \frac{2}{c}|v|,$$

so $f \circ i_a$ is conformal with constant conformal factor, hence is a similarity by Case 1. Thus $f = (f \circ i_a) \circ i_a^{-1}$ is the composition of an inversion and a similarity. ■

When the dimension n is two, the conclusion of Liouville's theorem is false: the Riemann mapping theorem asserts that the open unit disk admits a conformal map onto any simply connected open proper subset $U \subset \mathbb{R}^2$, while the image of the disk under a similarity or inversion is a disk or a half-plane.

This dichotomy depending on the dimension can be understood in the context of the proof above: the second equation $\lambda_i\lambda_j = \lambda_{ij}$ in (5) holds only when $n \geq 3$, because only then is the last sum in the preceding equation nonempty. Further, when $n = 2$, the first equation in (5) only asserts $\Delta\lambda = \lambda_{ii} + \lambda_{jj} = 0$, in other words that λ is just a harmonic function.

Nevertheless, a weaker alternative that includes the case $n = 2$ can be formulated:

Proposition 4. *Let $f : U \to \mathbb{R}^n$, ($n \geq 1$) be a conformal transformation of class C^2, with $U \subset \mathbb{R}^n$ a connected open set. If the conformal factor is constant, then f is a similarity.*

Proof. Since the conformal factor is constant, all partial derivatives of λ are zero, hence all second partial derivatives of f are zero by (4). Thus, each f_i is constant, so f is an affine linear transformation. It is then easy to see f is a similarity. ■

Remark 5. With g the Euclidean metric, the Christoffel symbols of (U, f^*g) for the Levi-Civita connection, computed in the standard Euclidean coordinate frame on U, are given by $\Gamma^k_{ij} = \lambda_i\delta_{jk} + \lambda_j\delta_{ki} - \lambda_k\delta_{ij}$, so that $f_{ij} = \sum_k \Gamma^k_{ij} f_k$ is equivalent to (4). The Riemann curvature endomorphism R of (U, f^*g) can then be shown to satisfy

$$R(\partial_i, \partial_j)\partial_k = f_{kji} - f_{kij} = 0.$$

The proof of Liouville's Theorem here—which derives the crucial equations (5) and (6) by expanding the identity $f_{iji} - f_{iij} = 0$ in terms of λ—can thus be remembered as a corollary of the flatness of (U, f^*g).

Other proofs of Liouville's theorem [**5, 6**] derive equations similar to (5) which result from the vanishing of the Riemann, Ricci, and Scalar curvatures of (U, f^*g). Again, the restrictions Liouville's theorem places on conformal maps can be regarded as consequences of the vanishing of these curvatures.

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MATHEW KUSHELMAN *Department of Mathematics, North Carolina State University, Raleigh NC 27695*
mdkushel@ncsu.edu

PETER MCGRATH *Department of Mathematics, North Carolina State University, Raleigh NC 27695*
pjmcgrat@ncsu.edu