Newton's Shell Theorem via Archimedes's Hat Box and Single-Variable Calculus

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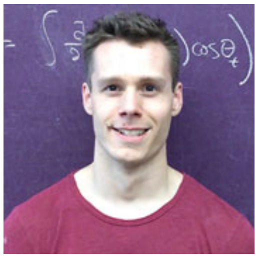

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Renate wants to share a chocolate-covered orange with you, with the caveat that your portion be selected in the following way: She will make two parallel cuts separated by a fixed distance d (which you do not get to pick) and will give you the piece of the orange in the slab in between—you get to pick where she makes the top cut. What height should you pick to maximize the surface area of chocolate on your piece?

Allow yourself time to ponder this before reading on, because the answer may challenge your intuition. If you have some knowledge of calculus, in particular if you are familiar with the formula for the area of a surface of revolution, then you might try to work the answer out for yourself (and compare your work with the calculation below!).

In an effort to put more space between here and the resolution of the problem (in case you want to think about it, but your eyes wander) and to put the problem on more rigorous footing, we make the following definition.

Definition. We call the subset of a sphere S in a slab between two parallel planes P_1 and P_2, each of which intersects S, a spherical segment.

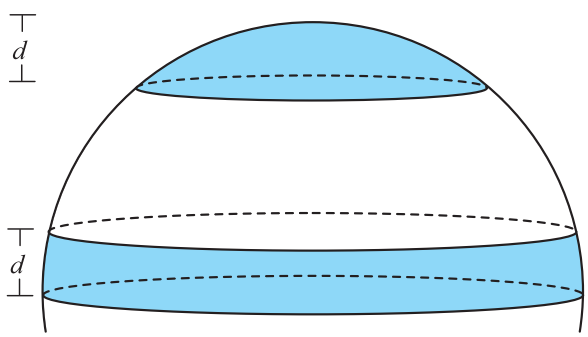

Figure 1. Which segment has larger area, the one at the pole or the one near the equator?

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Using this notation, we can rephrase our problem as follows: On a sphere of radius r, which spherical segments with vertical separation d have the largest surface area?

To suggest some subtlety to the problem, consider the following: In the extreme cases, the segment could be either a band around the equator or a cap at one of the poles, as shown in Figure 1. On one hand, the sphere is widest at the region surrounding the equator; on the other, while the sphere becomes narrower near the poles, it also becomes flatter. Which competing factor dominates?

This problem has been considered since antiquity and was resolved by Archimedes in what is known as his "hat box theorem" (we explain the name below). In the context of the thought experiment, it ends up that Renate is devious—your choice has no effect on the piece of chocolate's area! In other words, any two spherical segments with the same separation have equal areas. There is even a simple expression for the area.

Theorem 1 (Archimedes's hat box theorem). *The area of any spherical segment formed by a sphere of radius r and two planes separated by distance d is $2\pi rd$.*

The conclusion is that the competing factors discussed above, the relative breadth and flatness of the segment chosen, exactly balance. Perhaps you can see this cancellation in the following proof.

Proof. First, recall that the area of a surface given by rotating a graph $y = f(x)$ around the x-axis between $x = a$ and $x = b$ is

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

We can realize a sphere with radius r centered at the origin by rotating the function $f(x) = \sqrt{r^2 - x^2}$ defined on $-r \leq x \leq r$ about the x-axis. Calculating $f'(x)$ and working it into the area expression gives

$$2\pi \int_a^b \sqrt{r^2 - x^2}\, \frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi \int_a^b r\, dx = 2\pi r(b - a). \qquad \blacksquare$$

How did Archimedes prove this, without calculus? His ingenious argument shows that the area of a spherical segment formed by two planes with separation d is equal to the surface area cut from the cylinder of radius r enclosing the sphere whose axis is orthogonal to the planes. The theorem's name arises from imagining the cylinder as a box and the top half of the sphere as a hat inside. See [**1**] for an excellent exposition of this method.

A second thought experiment

Next, suppose that Mr. Z pilots a spaceship inside a planet-sized, hollow spherical shell with uniform mass density. How does the net force of the shell on him depend on his position? What if his ship is outside the shell?

Recall from elementary physics that the formula for the magnitude of the force F between two point particles separated by distance r and with masses m_1 and m_2 is

$$|F| = G\frac{m_1 m_2}{r^2} \tag{1}$$

where G is a gravitational constant.

This time, I will immediately let the cat out of the bag (or the hat out of the box?) and reveal the answer, which was discovered by Newton.

Theorem 2 (Newton's shell theorem). *Let p be a point particle with mass m and S be a thin spherical shell with radius r, uniform mass density, and total mass M. Let R be the distance between p and the center of S. The magnitude of the net gravitational force of S on p is*

$$|F_{net}| = \begin{cases} 0 & \text{if } R < r, \\ G\frac{Mm}{R^2} & \text{if } R > r. \end{cases}$$

Let us appreciate, as we did with Theorem 1, how this result is also independent of the geometric setup. In the language of the thought experiment above, the first part of Newton's shell theorem asserts that Mr. Z feels nothing from the shell, regardless of his position inside. As far as the net force is concerned, the shell may as well not exist!

The second part, the case where p is outside of S, asserts that the net gravitational force is the same as the situation where the entire shell S is replaced by a point mass M at its center. This part of Theorem 2 dramatically simplifies calculations in celestial mechanics because it allows planets to be replaced by point masses.

We begin with a heuristic understanding of the competing factors, as we did with Archimedes's result. Suppose p is inside S. If p is at the center of S, then the net force is zero by symmetry. But suppose p is very close to part of S, which we may think of as being composed of many point masses. On one hand, there are very large forces arising from point particles of S close to p due to the very small radii r in (1). On the other hand, there are oppositely directed forces arising from point particles on the other side of S. These forces are smaller in magnitude because the particles are further from p, but there are many more of these second type of points. The first part of Newton's result shows that there is a perfect net cancellation.

We can make the argument in the previous paragraph slightly more convincing without much more work: Fix p inside S and subtend from p an infinitesimal cone C which intersects S in caps C_1 and C_2 on opposite sides of S as shown in Figure 2. The points on C_1 are a fixed distance r_1 from p, hence by (1), the magnitude $|F_1|$ of the force from C_1 scales like r_1^{-2}. On the other hand, $|F_1|$ is proportional to the area of C_1, which scales like r_1^2. Combining these observations means $|F_1|$ is independent of r_1, and by using

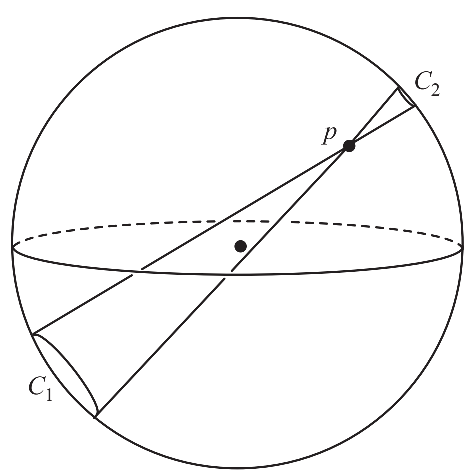

Figure 2. The forces from the caps C_1 and C_2 cancel.

the same argument to analyze the magnitude $|F_2|$ of the force from C_2, we see that $F_1 + F_2 = 0$, since C_1 and C_2 are on opposite sides of p.

Proving the shell theorem with the hat box theorem

We have observed qualitative similarities between Theorems 1 and 2, each asserting that a particular geometric quantity is independent of the initial setup. In fact, the connection runs deeper—below, we use the hat box theorem to prove the shell theorem.

A common proof of Theorem 2 evaluates the force from the shell as a triple integral in spherical coordinates [**2**, p. 40]. By symmetry considerations, it is possible to write the latter as a single, one-variable integral. In our case, Archimedes's theorem removes the necessity to set up this integral in angular coordinates, which would require invoking the law of cosines and a later application of a change of variables. We find that this makes the entire derivation more tractable and intuitive.

Proof of the shell theorem. After translating S and rotating p about S, we may suppose S is centered at zero and p is on the negative x-axis as shown in Figure 3. Because of this, symmetry implies that the y- and z-components F_y and F_z of the net force F_{net} vanish. Therefore, F_{net} is directed along the x-axis and satisfies $|F_{\text{net}}| = |F_x|$. We will compute $|F_{\text{net}}|$ by integrating the magnitudes of the forces from spherical segments of S with infinitesimal separation dx.

By (1), the magnitude $|dF|$ of the force between p and an infinitesimal piece dS of S with mass dM and distance d from p is

$$|dF| = \frac{Gm}{d^2}\, dM.$$

Since $F_y = F_z = 0$, we are interested only in the magnitude $|dF_x|$ of the x-components of such infinitesimal forces. From Figure 3 (remember that x is negative there), we see

$$|dF_x| = \frac{Gm}{d^2}\frac{R+x}{d}\, dM.$$

We now focus on spherical segments directed along the x-axis with infinitesimal thickness dx. Since S has constant mass density, the infinitesimal mass dM of such a segment is proportional to its area. By Theorem 1, all spherical segments with thickness dx have the same infinitesimal area, so $dM = c\, dx$ for some constant c. In fact, the constant $c = M/(2r)$, since $M = \int dM = \int_{-r}^{r} c\, dx = c(2r)$.

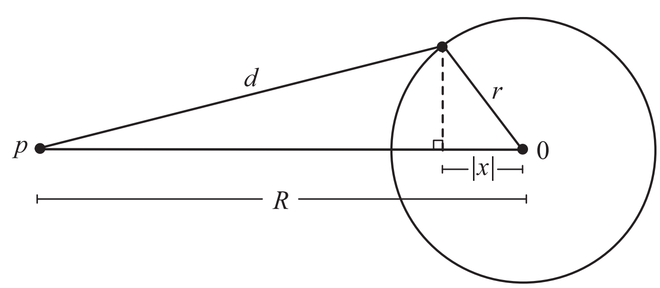

Figure 3. Setup for the shell theorem.

Integrating the forces $|dF_x|$ from these segments on $[-r, r]$, we find

$$|F_{\text{net}}| = \frac{GmM}{2r} \int_{-r}^{r} \frac{R+x}{d^3}\, dx.$$

Using the Pythagorean theorem twice (or the law of cosines), we have

$$d^2 = R^2 + r^2 + 2Rx,$$

hence

$$|F_{\text{net}}| = \frac{GmM}{2r} \int_{-r}^{r} \frac{R+x}{(R^2 + r^2 + 2Rx)^{\frac{3}{2}}}\, dx.$$

Take a deep breath; the intricate setup is done and the upshot is that we have reduced the entire problem to a single one-variable integral.

We integrate by parts, taking $u = R + x$ and C:

$$\begin{aligned}
\int \frac{R+x}{(R^2 + r^2 + 2Rx)^{\frac{3}{2}}}\, dx &= -\frac{R+x}{R(R^2 + r^2 + 2Rx)^{\frac{1}{2}}} - \int \frac{dx}{R(R^2 + r^2 + 2Rx)^{\frac{1}{2}}} \\
&= -\frac{R+x}{R\sqrt{R^2 + r^2 + 2Rx}} + \frac{\sqrt{R^2 + r^2 + 2Rx}}{R^2} \\
&= \frac{r^2 + Rx}{R^2\sqrt{R^2 + r^2 + 2Rx}}.
\end{aligned}$$

Returning to the original integral and introducing the appropriate bounds, we have

$$|F_{\text{net}}| = \frac{GmM}{2r} \left. \frac{r^2 + Rx}{R^2\sqrt{R^2 + r^2 + 2Rx}} \right|_{-r}^{r} = \frac{GmM}{2R^2} \left[\frac{r+R}{\sqrt{(r+R)^2}} + \frac{R-r}{\sqrt{(r-R)^2}} \right].$$

We now interpret this answer in the two cases.

If $R < r$, so that p is inside the shell, then the second term in the brackets is -1 and $|F_{\text{net}}| = 0$.

If $R > r$, so that p is outside the shell, then the second term in the brackets is 1 and $|F_{\text{net}}| = GMm/R^2$. ■

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Summary. Newton's shell theorem asserts that the net gravitational force between a point particle and a sphere with uniform mass density is the same as the force in the situation where the sphere is replaced by a point particle at its center with the same total mass. We give an exposition of this theorem using only tools from introductory one-variable calculus. A key simplification is a result of Archimedes that the area of the region on a sphere between two parallel planes depends only on the separation between the planes, not on their position relative to the sphere.

References

[1] Apostol, T., Mnatsakanian, M. (2004). A fresh look at the method of Archimedes. *Am. Math. Monthly* 111: 496–508. doi.org/10.2307/4145068.

[2] Menzel, D. H. (1961). *Mathematical Physics*. New York, NY: Dover.