TWO-POINT FUNCTIONS AND CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{R}^3$

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Abstract. Using a two-point maximum principle technique inspired by work of Brendle and Andrews-Li, we give a new proof of a special case of Alexandrov's theorem: that there are no embedded constant mean curvature tori in Euclidean three-space.

1. Introduction

Background and main results.

Constant mean curvature (CMC) surfaces are widely studied in geometric analysis, and there are still many important problems pertaining to their existence and classification. A fundamental result of Alexandrov [1] asserts that the only closed and embedded CMC hypersurfaces in $\mathbb{R}^n$ are the round spheres. Wente showed that the embeddedness hypothesis in Alexandrov's theorem is necessary by constructing [16] examples of immersed CMC tori in $\mathbb{R}^3$, and later Kapouleas constructed [10, 11] immersed CMC surfaces in $\mathbb{R}^3$ of each genus greater than one.

Alexandrov's proof introduced an ingenious technique now known as the method of moving planes, which has become an important tool in geometry and analysis [9, 14, 12]. This method applies the maximum principle to pieces of a surface that have been reflected about the one-parameter family of hyperplanes in $\mathbb{R}^n$ normal to a prescribed axis.

In the last decade, a number of important problems in geometry and analysis have been solved [3, 6, 4] by applying the maximum principle to a function depending on pairs of points of the underlying space. Among these were Brendle's characterization [6] of the Clifford torus as the only embedded minimal torus in the round three-sphere $\mathbb{S}^3$, and later Andrews-Li's classification [4] of the embedded CMC tori in $\mathbb{S}^3$ by their rotational symmetry. For more information about recent applications of two-point maximum principles, we refer to the survey articles [2, 7].

It is natural to wonder whether Brendle's approach and its adaptation by Andrews-Li applies to the setting of CMC tori in $\mathbb{R}^3$. The main result of this paper is to prove the following special case of Alexandrov's theorem using a two-point maximum principle:

Theorem 1.1. *There are no embedded CMC tori in $\mathbb{R}^3$.*

Using the same techniques, we prove the following in the noncompact setting.

Theorem 1.2. *Let $F : \Sigma \to \mathbb{R}^3$ be a CMC embedding with no umbilic points. If $M := F(\Sigma)$ is complete and singly periodic with compact quotient, then M is rotationally symmetric.*

Theorem 1.2 is a special case of a result [12, Theorem 2.10] of Korevaar-Kusner-Solomon—proved using the method of moving planes—which asserts that a properly embedded CMC surface contained in a solid cylinder is rotationally symmetric with respect to the axis of the cylinder.

The conclusions of Theorems 1.1 and 1.2 still hold if M is assumed to be only Alexandrov immersed instead of embedded, and the proofs given here could be appropriately modified in the spirit of [5] to apply to Alexandrov immersions.

Outline of the methods.

While the methods in this paper closely follow those in [4] and [6], we summarize the strategy for completeness. In Section 3, we introduce the *inscribed radius* at a point p on a closed and embedded surface $\Sigma \subset \mathbb{R}^3$, which is the radius of the largest ball tangent to p and contained in the region bounded by Σ. By using local information at p, it is clear that the inscribed radius must be at most the reciprocal $1/\lambda_1$ of the larger principal curvature λ_1. While in general, the inscribed radius may be smaller (see Figure 1), we show in Theorem 4.1 that the inscribed radius on an embedded CMC torus is everywhere equal to $1/\lambda_1$.

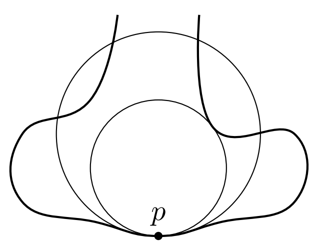

FIGURE 1. A schematic of a surface Σ whose inscribed radius at $p \in \Sigma$—the radius of the smaller ball—is less than $1/\lambda_1$ (the radius of the larger ball).

The proof of Theorem 4.1 involves applying the maximum principle to a geometrically motivated function $Z : \Sigma \times \Sigma \to \mathbb{R}$. Z depends on a smooth positive function $\Phi : \Sigma \to \mathbb{R}$, and $Z(x, \cdot)$ is everywhere nonnegative precisely when (see Proposition 3.2) the inscribed radius of Σ at x is at least $1/\Phi(x)$. It turns out that at a critical point, Z satisfies a PDE (see Proposition 3.13) related to the Simons equation satisfied by the norm squared of the second fundamental form h. Using these equations in conjunction with the maximum principle applied to Z proves that $\Phi = 1/\lambda_1$, and Theorem 4.1 follows.

To prove Theorem 1.1, we consider an orthonormal frame $\{e_1, e_2\}$ for Σ and show using Theorem 4.1 that h is parallel in the e_1 direction. This suggests that M is rotationally symmetric, and we show in Lemma 4.8 that the axis L orthogonal to both e_1 and the acceleration vector field $\overline{\nabla}_{e_1} e_1$ is constant on Σ. We then find a point $p \in \Sigma$ with $T_p\Sigma = L^\perp$, and it follows that p is an umbilic point of Σ, which contradicts [8].

The hypotheses of Theorem 1.2 allow us to conclude in essentially the same way as in the setting of Theorem 1.1 that the line L above is constant on Σ, and we prove Theorem 1.2 by constructing a rotationally symmetric parametrization for Σ with L as its axis.

2. Preliminaries

Notation and basic definitions.

Let Σ be a closed surface, $F : \Sigma \to \mathbb{R}^3$ be an embedding, and ν be the unit outward pointing normal field on $M := F(\Sigma)$. Let g denote the metric on M induced by the Euclidean metric δ_{ij} on $\mathbb{R}^3$. We denote the Levi-Civita connection on $\mathbb{R}^3$ induced by δ_{ij} by $\overline{\nabla}$ and the Levi-Civita connection on M induced by g by ∇.

The *shape operator* is the endomorphism defined by $B(w) = \overline{\nabla}_w \nu$, where w is a tangent vector on M. Its eigenvalues are called the *principal curvatures* and will be denoted λ_1, λ_2, where $\lambda_1 \geq \lambda_2$. Then, the *scalar second fundamental form on M* is the symmetric two-tensor defined by $h(v, w) = \langle B(v), w\rangle = \langle \overline{\nabla}_v \nu, w\rangle = -\langle \overline{\nabla}_v w, \nu\rangle$. The *mean curvature* of M is $H = \frac{\lambda_1+\lambda_2}{2}$, and the *trace-free second fundamental form* on M is the symmetric two-tensor $\mathring{h}$ defined by $\mathring{h} = h - Hg$. Elementary computations show that $|\mathring{h}|^2 = |h|^2 - 2H^2$ and $\lambda_1\lambda_2 = 2H^2 - \frac{|h|^2}{2}$.

The *Riemann curvature tensor* on M is defined by

$$Rm(X, Y, Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\rangle,$$

and the *Gauss equation* asserts that $R_{ijkm} = h_{jk}h_{im} - h_{ik}h_{jm}$, where R_{ijkm} and h_{jk} denote components of the curvature tensor and of the second fundamental form in any local frame $\{E_i\}$. The *Codazzi equation* states that

$$(\nabla_X h)(Y, Z) = (\nabla_Z h)(Y, X), \quad \text{equivalently} \quad h_{ij;k} = h_{kj;i},$$

where the indices after the semicolon are the ones corresponding to covariant differentiation. We also recall the *Ricci identity*:

$$h_{ij;kl} = h_{ij;lk} + \sum_m R_{kjim}h_{mj} + \sum_m R_{lkjm}h_{mi}. \tag{2.1}$$

Lastly, we let h_i^m denote the m^{th} component of $B(E_i)$, and we define $g_{ij} = g(E_i, E_j)$. These can be shown to satisfy

$$h_{ij} = \sum_m g_{im}h_j^m. \tag{2.2}$$

The Simons equation.

The following partial differential equations concerning the norm squared of the second fundamental form of a CMC immersion are well-known [15, 13], although we provide a proof for completeness.

Proposition 2.3. *Suppose that $F : \Sigma \to \mathbb{R}^3$ is a CMC immersion. Then*

(i) $\frac{1}{2}\Delta|h|^2 = |\nabla h|^2 - |h|^4 + 6|h|^2H^2 - 8H^4$.

(ii) $\frac{1}{2}\Delta|\mathring{h}|^2 = |\nabla h|^2 - |\mathring{h}|^4 + 2|\mathring{h}|^2H^2$.

(iii) *Whenever $\mu := |\mathring{h}|/\sqrt{2} = \lambda_1 - H = \frac{1}{2}(\lambda_1 - \lambda_2)$ is nonzero,*

$$\Delta\mu - \frac{|\nabla\mu|^2}{\mu} + 2(\mu^2 - H^2)\mu = 0.$$

Proof. To prove (i), let $x \in \Sigma$, and choose normal coordinates about x. Calculating and using the Codazzi equation, we find that

$$\begin{aligned}\frac{1}{2}\Delta|h|^2 &= \sum_{i,j}\frac{1}{2}\Delta h_{ij}^2 \\ &= \sum_{i,j,k} h_{ij}h_{ij;kk} + |\nabla h|^2 \\ &= \sum_{i,j,k} h_{ij}h_{ik;jk} + |\nabla h|^2.\end{aligned}$$

From the preceding, (2.1), and the Gauss equation, we find

$$\begin{aligned}\frac{1}{2}\Delta|h|^2 &= \sum_{i,j,k} h_{ij}h_{ik;kj} + \sum_{i,j,k,m} h_{ij}R_{kjmi}h_{mk} + \sum_{i,j,k,m} h_{ij}R_{kjmk}h_{mi} + |\nabla h|^2 \\ &= \sum_{i,j,k} h_{ij}h_{kk;ij} + \sum_{i,j,k,m} h_{ij}(h_{ki}h_{jm} - h_{ji}h_{km})h_{mk} \\ &\quad + \sum_{i,j,k,m} h_{ij}(h_{kk}h_{jm} - h_{jk}h_{km})h_{mi} + |\nabla h|^2 \\ &= \sum_{i,j} h_{ij}(2H)_{;ij} + \sum_{i,j,k,m} h_{ij}h_{ki}h_{jm}h_{mk} - \sum_{i,j,k,m} h_{ij}h_{ji}h_{km}h_{mk} \\ &\quad + \sum_{i,j,k,m} h_{ij}h_{kk}h_{jm}h_{mi} - \sum_{i,j,k,m} h_{ij}h_{jk}h_{km}h_{mi} + |\nabla h|^2.\end{aligned}$$

Since H is constant and the second and fifth sums above cancel, we find

$$\begin{aligned}\frac{1}{2}\Delta|h|^2 &= -\sum_{i,j,k,m} h_{ij}h_{ji}h_{km}h_{mk} + \sum_{i,j,k,m} h_{ij}h_{kk}h_{jm}h_{mi} + |\nabla h|^2 \\ &= -|h|^4 + 2H\sum_{i,j,m} h_{ij}h_{jm}h_{mi} + |\nabla h|^2.\end{aligned}$$

We further compute that

$$\begin{aligned}\sum_{i,j,m} h_{ij}h_{jm}h_{mi} &= \lambda_1^3 + \lambda_2^3 \\ &= (\lambda_1+\lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1+\lambda_2) \\ &= 8H^3 - 3(2H^2 - \frac{1}{2}|h|^2)(2H) \\ &= -4H^3 + 3|h|^2 H.\end{aligned}$$

Combining this with the preceding proves (i). Item (ii) follows directly from using (i), that $|h|^2 = |\mathring{h}|^2 + 2H^2$, and that H is constant so $\Delta H^2 = 0$.

Next, consider a point where $|\mathring{h}|$ is nonzero. By a straightforward calculation using the Codazzi equations and that H is constant, we have that $|\nabla h|^2 = 2|\nabla|\mathring{h}||^2$, and therefore from item (ii) that

$$\frac{1}{2}\Delta|\mathring{h}|^2 = 2|\nabla|\mathring{h}||^2 - |\mathring{h}|^4 + 2|\mathring{h}|^2 H^2.$$

Because $|\mathring{h}|^2 = 2\mu^2$, we conclude that

$$\Delta\mu^2 = 4|\nabla\mu|^2 - 4\mu^4 + 4\mu^2 H^2. \tag{2.4}$$

Since $\Delta\mu^2 = 2\mu\Delta\mu + 2|\nabla\mu|^2$, item (iii) follows by rearranging (2.4). □

3. The Inscribed Radius and the Two-Point Function

Suppose now that Σ is a closed surface and that $F : \Sigma \to \mathbb{R}^3$ is an embedding. Since $M := F(\Sigma)$ is closed and embedded, it bounds a precompact region $\Omega \subset \mathbb{R}^3$. We call the radius of the largest ball contained in $\overline{\Omega}$ and tangent to M at $F(x)$ the *inscribed radius* of M at $F(x)$.

Now fix a positive smooth function $\Phi : \Sigma \to \mathbb{R}$. We define a function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x,y) = \frac{\Phi(x)}{2}|F(y) - F(x)|^2 + \langle F(y) - F(x), \nu(x)\rangle. \tag{3.1}$$

Proposition 3.2. *Z is everywhere non-negative if and only if for every $x \in \Sigma$, the inscribed radius of M at $F(x)$ is at least $1/\Phi(x)$.*

Proof. Let $x \in \Sigma$ and consider the ball B with center $F(x) - \frac{1}{\Phi(x)}\nu(x)$ and radius $1/\Phi(x)$. B is tangent to M at $F(x)$, and the condition that $B \subset \overline{\Omega}$ is equivalent to the condition that

$$\left|F(y) - \left(F(x) - \frac{1}{\Phi(x)}\nu(x)\right)\right|^2 \geq \frac{1}{\Phi^2(x)} \quad \forall y \in \Sigma. \tag{3.3}$$

By expanding, we see that (3.3) is equivalent to $\frac{2}{\Phi(x)}Z(x,y) \geq 0$ for all $y \in \Sigma$, which implies the conclusion. □

We now assume that M has constant mean curvature H. Since M is closed, our earlier conventions imply that $H \geq 0$. Moreover, since there are no closed embedded minimal surfaces in $\mathbb{R}^3$, we may assume that $H > 0$.

Assumption 3.4. We assume that Z as in (3.1) is everywhere non-negative, and that $Z(\overline{x}, \overline{y}) = 0$ for some pair of distinct points $\overline{x}, \overline{y} \in \Sigma$.

Note that the differential of Z vanishes at $(\overline{x}, \overline{y})$. Moreover, the reflection $R : \mathbb{R}^3 \to \mathbb{R}^3$ about the planar subspace orthogonal to $F(\overline{y}) - F(\overline{x})$ given by

$$R(z) = z - 2\left\langle \frac{F(\overline{y}) - F(\overline{x})}{|F(\overline{y}) - F(\overline{x})|}, z \right\rangle \frac{F(\overline{y}) - F(\overline{x})}{|F(\overline{y}) - F(\overline{x})|} \tag{3.5}$$

maps the tangent space to M at $\overline{x}$ to the tangent space of M at $\overline{y}$ (See Figure 2).

Choice of Coordinates.
We now choose convenient coordinate systems about $F(\overline{x})$ and $F(\overline{y})$, which we sometimes write as $\overline{x}$ and $\overline{y}$ respectively to make notation simpler. About $F(\overline{x})$ we choose normal coordinates (x^1, x^2) diagonalizing the second fundamental form at $F(\overline{x})$, so that

$$h\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) = \lambda_i \delta_{ij} \quad \text{at} \quad F(\overline{x}). \tag{3.6}$$

Moreover, at $F(\overline{x})$, (2.2) becomes

$$h_{ij} = \sum_m g_{im} h_j^m = \sum_m \delta_{im} h_m^j = h_i^j. \tag{3.7}$$

About $F(\overline{y})$ we take normal coordinates (y^1, y^2) whose coordinate vectors are the images under R of the corresponding coordinate vectors at $F(\overline{x})$, that is $\frac{\partial F}{\partial y^i}(\overline{y}) = R\left(\frac{\partial F}{\partial x^i}(\overline{x})\right)$ as seen in Figure 2.

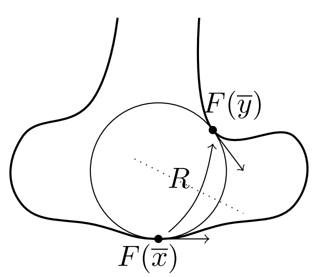

FIGURE 2. A schematic of how the reflection R is used to make convenient coordinates at $F(\overline{y})$.

Lemma 3.8. *The following equations hold:*

$$\langle F(\overline{y}) - F(\overline{x}), \nu(\overline{x}) \rangle = -\frac{\Phi(\overline{x})}{2} |F(\overline{y}) - F(\overline{x})|^2 \tag{3.9}$$

$$\left\langle F(\overline{y}) - F(\overline{x}), \frac{\partial F}{\partial x^i}(\overline{x}) \right\rangle = \frac{|F(\overline{y}) - F(\overline{x})|^2}{2} \frac{\frac{\partial \Phi}{\partial x^i}(\overline{x})}{\Phi(\overline{x}) - \lambda_i(\overline{x})} \tag{3.10}$$

$$\frac{\partial F}{\partial y^i}(\overline{y}) - \frac{\partial F}{\partial x^i}(\overline{x}) = -\frac{\frac{\partial \Phi}{\partial x^i}(\overline{x})}{\Phi(\overline{x}) - \lambda_i(\overline{x})}(F(\overline{y}) - F(\overline{x})) \tag{3.11}$$

$$\nu(\overline{y}) - \nu(\overline{x}) = \Phi(\overline{x})(F(\overline{y}) - F(\overline{x})). \tag{3.12}$$

Proof. Identity (3.9) follows from rearranging the equation $Z(\overline{x}, \overline{y}) = 0$ using (3.1). Equation (3.10) follows from computing

$$\begin{aligned} \frac{\partial Z}{\partial x^i} = \frac{1}{2}\frac{\partial \Phi}{\partial x^i}|F(y) - F(x)|^2 - \Phi(x)\left\langle F(y) - F(x), \frac{\partial F}{\partial x^i}\right\rangle \\ - \left\langle \frac{\partial F}{\partial x^i}, \nu(x)\right\rangle + \left\langle F(y) - F(x), \sum_p h_i^p(x)\frac{\partial F}{\partial x^p}\right\rangle \end{aligned}$$

and evaluating at $(\overline{x}, \overline{y})$ using (3.6), (3.7), and $\frac{\partial Z}{\partial x^i}(\overline{x}, \overline{y}) = 0$. Equation (3.11) follows from applying (3.5) to $z = \frac{\partial F}{\partial x^i}(\overline{x})$ and using (3.10). Equation (3.12) follows from applying (3.5) with $z = \nu(\overline{x})$ and using (3.9). □

Properties of the function Z.

Proposition 3.13. *If $\Phi(\overline{x}) > \lambda_1$, then at the point $(\overline{x}, \overline{y})$*

$$\begin{aligned} &\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z = \\ &= \frac{|F(\overline{y}) - F(\overline{x})|^2}{2}\left(\Delta\Phi - 2\sum_i \frac{\left(\frac{\partial \Phi}{\partial x^i}\right)^2}{\Phi(\overline{x}) - \lambda_i(\overline{x})} + (|h|^2(\overline{x}) - 2H\Phi(\overline{x}))\Phi(\overline{x})\right). \end{aligned}$$

Proof. Using (3.1), we compute

$$\begin{aligned} &\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right) Z = \\ &\quad = \frac{1}{2}\frac{\partial \Phi}{\partial x^i}|F(y) - F(x)|^2 + \Phi(x)\left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, F(y) - F(x)\right\rangle \\ &\quad + \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, \nu(x)\right\rangle + \left\langle F(y) - F(x), \sum_p h_i^{p(x)}\frac{\partial F}{\partial x^p}\right\rangle \\ &\quad = \frac{1}{2}\frac{\partial \Phi}{\partial x^i}|F(y) - F(x)|^2 + \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, \Phi(x)(F(y) - F(x)) + \nu(x)\right\rangle \\ &\quad + \sum_p h_i^p(x)\left\langle F(y) - F(x), \frac{\partial F}{\partial x^p}\right\rangle. \end{aligned}$$

We compute the second derivatives and sum over i to see that

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z =$$

$$= \frac{|F(y)-F(x)|^2}{2} \sum_i \frac{\partial^2 \Phi}{\partial (x^i)^2} + 2\sum_i \frac{\partial \Phi}{\partial x^i} \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, F(y)-F(x) \right\rangle$$

$$+ \sum_i \left\langle \frac{\partial^2 F}{\partial (y^i)^2} - \frac{\partial^2 F}{\partial (x^i)^2}, \Phi(x)(F(y)-F(x)) + \nu(x) \right\rangle$$

$$+ \sum_i \Phi(x) \left| \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i} \right|^2 + 2\sum_{i,p} h_i^p(x) \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^p} \right\rangle$$

$$+ \sum_{i,p} h_i^p(x) \left\langle F(y)-F(x), \frac{\partial^2 F}{\partial x^i \partial x^p} \right\rangle$$

$$+ \sum_{i,p} h^p_{i,i}(x) \left\langle F(y)-F(x), \frac{\partial F}{\partial x^p} \right\rangle .$$

At $\overline{x}$, we now remark that

$$\frac{\partial^2 F}{\partial (x^i)^2}(\overline{x}) = \left\langle \nabla_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^i}(\overline{x}), \nu(\overline{x}) \right\rangle \nu(\overline{x}) + \sum_j \left\langle \nabla_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^i}(\overline{x}), \frac{\partial F}{\partial x^j}(\overline{x}) \right\rangle \frac{\partial F}{\partial x^j}(\overline{x})$$

$$= -h_{ii}(\overline{x})\nu(\overline{x}) + \Gamma^j_{ii}(\overline{x}) \frac{\partial F}{\partial x^j}(\overline{x}) = -\lambda_i \nu(\overline{x}),$$

since all the Christoffel symbols vanish at $\overline{x}$ by our choice of coordinates. A similar equation holds true at $\overline{y}$. We also note that from our choice of coordinates and the Codazzi Equations, at $\overline{x}$ we obtain

$$\sum_i h^p_{i,i} = \sum_i h^p_{i;i} = \sum_i h_{ii;}{}^p = (2H)_;^p = 0.$$

Hence when we evaluate at $(\overline{x}, \overline{y})$ using (3.6), (3.7), and the above considerations, we find that

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z\Big|_{(\overline{x},\overline{y})} =$$

$$= \frac{|F(\overline{y})-F(\overline{x})|^2}{2} \Delta\Phi + 2\sum_i \frac{\partial \Phi}{\partial x^i} \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, F(\overline{y})-F(\overline{x}) \right\rangle$$

$$- 2H \left\langle \nu(\overline{y}) - \nu(\overline{x}), \Phi(\overline{x})(F(\overline{y})-F(\overline{x})) + \nu(\overline{x}) \right\rangle + \sum_i \Phi(\overline{x}) \left| \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i} \right|^2$$

$$+ 2\sum_i \lambda_i(\overline{x}) \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^i} \right\rangle - |h|^2(\overline{x}) \left\langle F(\overline{y})-F(\overline{x}), \nu(\overline{x}) \right\rangle .$$

The conclusion now follows from substituting using Lemma 3.8 and simplifying. □

Corollary 3.14. *If $\Phi(\overline{x}) > \lambda_1$, then at $(\overline{x}, \overline{y})$ we have*

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z \leq \frac{|F(\overline{y}) - F(\overline{x})|^2}{2}\left(\Delta\Phi - \frac{|\nabla\Phi|^2}{\Phi - H} + (|h|^2 - 2H\Phi)\Phi\right).$$

Proof. The result follows from combining Proposition 3.13 with the estimates

$$\Phi - \lambda_2 = \Phi + \lambda_1 - 2H \leq 2(\Phi - H) \quad \text{and} \quad \Phi - \lambda_1 \leq 2(\Phi - H).$$

□

4. Proof of Theorem 1.1

We now assume that Σ is a torus and that $F : \Sigma \to \mathbb{R}^3$ is a CMC embedding. Since M has no umbilical points [8], we can define a smooth orthonormal frame $\{e_1, e_2\}$ on M satisfying

$$h(e_1, e_1) = \lambda_1, \quad h(e_1, e_2) = 0, \quad \text{and} \quad h(e_2, e_2) = \lambda_2.$$

We may also assume that $\{e_1, e_2, \nu\}$ is a positively oriented frame for $\mathbb{R}^3$.

Theorem 4.1. *If Σ is a torus and $F : \Sigma \to \mathbb{R}^3$ is a CMC embedding, then for all $x \in \Sigma$, the inscribed radius at $F(x)$ is equal to $1/\lambda_1(x)$.*

Proof. We follow the outline of the proof of Theorem 5 in [4]. Let $x \in \Sigma$ and γ a geodesic on Σ satisfying $\gamma(0) = x$. It follows that

$$\frac{2\langle F(\gamma(s)) - F(x), \nu(x)\rangle}{s^2} = -h_x(\dot\gamma, \dot\gamma) + O(s).$$

Now fix a positive constant κ and define $\Phi = \kappa\mu + H$, where $\mu := \lambda_1 - H$ is as in Proposition 2.3(iii). It follows that

$$Z(x, \gamma(s)) = \frac{1}{2}\left(\kappa\mu + H - h_x(\dot\gamma, \dot\gamma)\right) s^2 + O(s^3). \tag{4.2}$$

By using (4.2) with a geodesic γ satisfying $\dot\gamma(0) = e_1$, and by noticing that $h(\dot\gamma, \dot\gamma) = \lambda_1 = \mu + H$, we obtain that

$$Z(x, \gamma(s)) = \frac{1}{2}(\kappa - 1)\mu s^2 + O(s^3). \tag{4.3}$$

Hence for any $\kappa < 1$, Z takes on negative values near x. Thus by Proposition 3.2 the inscribed radius can not be more that $1/\lambda_1$ for any $x \in \Sigma$. Hence it suffices to consider $\kappa \geq 1$.

For the remainder of this argument, we define Z_κ to be the function Z as in (3.1) with the choice of $\Phi = \kappa\mu + H$. We then define

$$K := \{\kappa > 0 : Z_\kappa \geq 0\} \quad \text{and} \quad \underline{\kappa} := \inf K.$$

First we claim that $Z_{\underline{\kappa}} \geq 0$. If not, then there are $x, y \in \Sigma$ such that $Z_\kappa(x, y) < 0$. Since the assignment $\kappa \to Z_\kappa(x, y)$ is continuous, there must exist a $\kappa' \in K$ such that $Z_{\kappa'}(x,y) < 0$, which is a contradiction. Hence $\underline{\kappa} \in K$, and we also see that if $\underline{\kappa} = 1$, we are done.

Suppose for the sake of a contradiction that $\underline{\kappa} > 1$. By (4.3), there is a neighborhood U of the diagonal $\{(x,x) : x \in \Sigma\}$ such that $Z_\kappa > 0$ on $U \setminus \{(x,x) : x \in \Sigma\}$ for all κ sufficiently close to $\underline{\kappa}$.

We now claim that there exists a point $(\overline{x}, \overline{y})$ with $\overline{x} \neq \overline{y}$ satisfying $Z_{\underline{\kappa}}(\overline{x}, \overline{y}) = 0$. If this were not the case, then $Z_{\underline{\kappa}}$ would attain a positive minimum on the compact set $\Sigma \times \Sigma \setminus U$, and using the continuous dependence of Z_κ on κ, it would follow that Z_κ is nonnegative on $\Sigma \times \Sigma$ for κ close to, but strictly less than $\underline{\kappa}$. This contradiction of the minimality of $\underline{\kappa}$ proves the claim.

Since $(\overline{x}, \overline{y})$ minimizes $Z_{\underline{\kappa}}$,

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right)^2 Z_{\underline{\kappa}} \Big|_{(\overline{x}, \overline{y})} \geq 0. \tag{4.4}$$

On the other hand, the preceding shows that Assumption 3.4 holds. Since Σ has no umbilic points by [8], by combining Corollary 3.14 with Proposition 2.3(iii), we find that

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right)^2 Z_{\underline{\kappa}} \Big|_{(\overline{x}, \overline{y})} \leq -(\underline{\kappa}^2 - 1)|F(\overline{y}) - F(\overline{x})|^2 \mu^2 H < 0.$$

This contradicts (4.4) and concludes the proof. □

Below, we repeatedly use that the Christoffel symbols $\Gamma_{ij}^k := \langle \nabla_{e_i} e_j, e_k \rangle$ in the frame $\{e_1, e_2\}$ satisfy $\Gamma_{ij}^k = -\Gamma_{ik}^j$, and also for all $i, j, k \in \{1, 2\}$ that

$$e_k(h(e_i, e_j)) = (\nabla_{e_k} h)(e_i, e_j) + \Gamma_{ki}^m h(e_m, e_j) + \Gamma_{kj}^m h(e_i, e_m). \tag{4.5}$$

Lemma 4.6. *We have that $(\nabla_{e_1} h)(e_1, e_1) = 0$ everywhere.*

Proof. For simplicity, we identify Σ with its image under the embedding F. Fix a point $x \in \Sigma$ and consider the geodesic $\gamma(t) = \exp_x^\Sigma(t e_1(x))$. By Proposition 3.2 and Theorem 4.1, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \frac{\lambda_1}{2} |\gamma(t) - x|^2 + \langle \gamma(t) - x, \nu(x) \rangle$$

is everywhere nonnegative. We calculate

$$\begin{aligned} f'(t) &= \langle \dot\gamma(t), \lambda_1 \gamma(t) - \lambda_1 x + \nu(x) \rangle, \\ f''(t) &= \lambda_1 - h(\dot\gamma(t), \dot\gamma(t)) \langle \nu(\gamma(t)), \lambda_1 \gamma(t) - \lambda_1 x + \nu(x) \rangle, \\ f'''(t) &= -(\overline{\nabla}_{\dot\gamma(t)} h)(\dot\gamma(t), \dot\gamma(t)) \langle \nu(\gamma(t)), \lambda_1 \gamma(t) - \lambda_1 x + \nu(x) \rangle \\ &\quad - h(\gamma(t), \gamma(t)) \langle \overline{\nabla}_{\dot\gamma(t)} \nu, \lambda_1 \gamma(t) - \lambda_1 x + \nu(x) \rangle, \end{aligned}$$

from which it follows easily that $f(0) = f'(0) = f''(0) = 0$. Since f is nonnegative, it follows that $0 = f'''(0) = (\nabla_{e_1} h)(e_1, e_1)$. □

Corollary 4.7. *The Christoffel symbols with respect to the frame* $\{e_1, e_2\}$ *satisfy* $\Gamma^2_{11} = \frac{e_2(\mu)}{2\mu}$ *and* $\Gamma^2_{21} = 0$. *Equivalently,*

$$\nabla_{e_1} e_1 = \frac{e_2(\mu)}{2\mu} e_2, \qquad \nabla_{e_2} e_1 = 0,$$
$$\nabla_{e_1} e_2 = -\frac{e_2(\mu)}{2\mu} e_1, \qquad \nabla_{e_2} e_2 = 0.$$

Proof. The equivalence of the statements above is clear from the symmetries of the Christoffel symbols. Considering the instances of (4.5) when (i, j, k) are the ordered three-tuples $(1, 2, 1)$, $(1, 1, 2)$, and $(1, 2, 2)$ and simplifying gives

$$\begin{aligned}
(\nabla_{e_1} h)(e_1, e_2) &= (\lambda_1 - \lambda_2)\Gamma^2_{11} = 2\mu\Gamma^2_{11},\\
(\nabla_{e_2} h)(e_1, e_1) &= e_2(\lambda_1) = e_2(\mu),\\
(\nabla_{e_2} h)(e_1, e_2) &= 2\mu\Gamma^2_{21}.
\end{aligned}$$

The first two of these equations and the Codazzi equations imply $\Gamma^2_{11} = \frac{e_2(\mu)}{2\mu}$. By combining Lemma 4.6, the constant mean curvature condition, and the Codazzi equations, we have $0 = (\nabla_{e_1} h)(e_1, e_1) = (\nabla_{e_2} h)(e_1, e_2)$, which by the third equation above implies $\Gamma^2_{21} = 0$. ☐

Lemma 4.8. *The line* L *spanned by* $e_1 \times \overline{\nabla}_{e_1} e_1$ *is constant on* M.

Proof. Since M is connected, it suffices to show that the ambient covariant derivatives of the vector field $e_1 \times \overline{\nabla}_{e_1} e_1 = \lambda_1 e_2 + \frac{e_2(\mu)}{2\mu}\nu$ are contained in L, where the equality follows by computing the cross product and using Corollary 4.7 to compute $\overline{\nabla}_{e_1} e_1 = \frac{e_2(\mu)}{2\mu} e_2 - \lambda_1 \nu$.

We first claim that $e_1(e_2(\mu)) = h_{11,21} = 0$. To see this, note first that by a calculation using Lemma 4.6 and Corollary 4.7, it follows that $h_{11,21} = h_{11;21}$ and $h_{11;12} = 0$. On the other hand, the Ricci identity (2.1), the symmetries of the curvature tensor, and our choice of frame imply that $h_{11;21} = h_{11;12}$, which proves the claim. Now, using the claim, and Lemma 4.6, we obtain

$$\begin{aligned}
\overline{\nabla}_{e_1}\left(\lambda_1 e_2 + \frac{e_2(\mu)}{2\mu}\nu\right) &= \lambda_1 \overline{\nabla}_{e_1} e_2 + \frac{e_2(\mu)}{2\mu}\overline{\nabla}_{e_1}\nu\\
&= 0,
\end{aligned}$$

where the second equality uses Corollary 4.7. Next, the Gauss Equation and and Corollary 4.7 imply that

$$\begin{aligned}
\lambda_1\lambda_2 &= R_{1221}\\
&= \left\langle \nabla_{e_1}\nabla_{e_2} e_2 - \nabla_{e_2}\nabla_{e_1} e_2 - \nabla_{[e_1,e_2]} e_2, e_1 \right\rangle\\
&= e_2\left(\frac{e_2(\mu)}{2\mu}\right) - \left(\frac{e_2(\mu)}{2\mu}\right)^2.
\end{aligned} \tag{4.9}$$

By using Corollary 4.7, the Gauss Formula, and (4.9), we compute

$$\begin{aligned}\overline{\nabla}_{e_2}\left(\lambda_1 e_2 + \frac{e_2(\mu)}{2\mu}\nu\right) &= \lambda_1 \overline{\nabla}_{e_2} e_2 + e_2(\lambda_1)e_2 + \frac{e_2(\mu)}{2\mu}\overline{\nabla}_{e_2}\nu + e_2\left(\frac{e_2(\mu)}{2\mu}\right)\nu \\ &= -\lambda_1\lambda_2\nu + e_2(\mu)e_2 + \frac{e_2(\mu)}{2\mu}\lambda_2 e_2 + e_2\left(\frac{e_2(\mu)}{2\mu}\right)\nu \\ &= e_2(\mu)\left(1 + \frac{\lambda_2}{2\mu}\right)e_2 + \left(\frac{e_2(\mu)}{2\mu}\right)^2\nu \\ &= \lambda_1\frac{e_2(\mu)}{2\mu}e_2 + \left(\frac{e_2(\mu)}{2\mu}\right)^2\nu \\ &= \frac{e_2(\mu)}{2\mu}\left(\lambda_1 e_2 + \frac{e_2(\mu)}{2\mu}\nu\right) \in L.\end{aligned}$$

□

Proof of Theorem 1.1. Because M is a closed surface, there exists a point $p \in M$ such that $\nu(p) \in L$, which by Lemma 4.8 implies that $\lambda_1(p) = 0$. Since $H \geq 0$, we have $\lambda_1(p) = \lambda_2(p) = 0$, so p is an umbilic point of M, which contradicts [8]. □

5. Rotational Symmetry in the Singly Periodic Case

We say that a set $S \subset \mathbb{R}^3$ is *singly periodic* if there exists a vector $v \in \mathbb{R}^3$ such that $v + S = S$. A singly periodic set has *compact quotient* if $S/\sim$ is compact, where $\sim$ is the equivalence relation on $\mathbb{R}^3$ defined by $x \sim y$ if and only if $x - y \in \mathbb{Z}v$. For the convenience of the reader, we restate Theorem 1.2.

Theorem 1.2. *Let $F : \Sigma \to \mathbb{R}^3$ be a CMC embedding with no umbilic points. If $M := F(\Sigma)$ is complete and singly periodic with compact quotient, then M is rotationally symmetric.*

In order to prove Theorem 1.2, we note that the hypotheses that M has no umbilic points and has compact quotient allow us to repeat the arguments in Section 4 with only superficial modifications. In particular, the conclusions of Lemma 4.6, Corollary 4.7, and Lemma 4.8 apply in the setting of Theorem 1.2. We will prove Theorem 1.2 by constructing in Proposition 5.8 a rotationally symmetric parametrization for M whose axis is the line L from Lemma 4.8.

Lemma 5.1. *The vector fields $E_1 := \frac{e_1}{\sqrt{\mu}}$ and $E_2 := e_2$ commute.*

Proof. By Corollary 4.7,

$$\left[\frac{e_1}{\sqrt{\mu}}, e_2\right] = \frac{1}{\sqrt{\mu}}\nabla_{e_1} e_2 - \nabla_{e_2}\frac{e_1}{\sqrt{\mu}} = -\frac{e_2(\mu)}{2\mu^{3/2}}e_1 + \frac{e_2(\mu)}{2\mu^{3/2}}e_1 = 0.$$

□

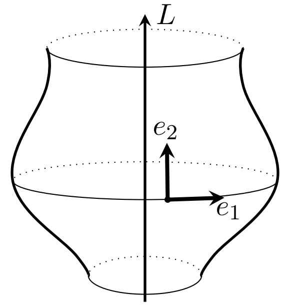

FIGURE 3. A sketch of part of M, with the frame $\{e_1, e_2\}$ labeled at a point. In Proposition 5.8 we characterize L as an axis of rotation of M.

By Lemma 5.1, there are local coordinates (s,t) on M whose coordinate vectors satisfy $(\partial/\partial s, \partial/\partial t) = (E_1, E_2)$ (see Figure 2). Then by Lemma 4.6, we see that λ_1, and hence μ, only depends on t.

Lemma 5.2. *The following equations hold, where* $w := \mu^{-\frac{1}{2}}$.

(i) $\frac{e_2(w)}{w} = -\frac{e_2(\mu)}{2\mu}$,
(ii) $\frac{e_2(e_2(w))}{w} + \lambda_1\lambda_2 = 0$,
(iii) *There exists a constant* $C \geq 4H$ *such that* $(e_2(w))^2 + \lambda_1^2 w^2 = C$.
(iv) $|\overline{\nabla}_{e_1} e_1| = \sqrt{C}/w$.

Proof. Item (i) is an immediate consequence of the definition $w = \mu^{-\frac{1}{2}}$.

Item (ii) follows by a calculation combining (i) with (4.9). Multiplying the equation in (ii) by the integrating factor $2we_2(w)$ and using the identity $\lambda_1\lambda_2 = H^2 - \mu^2 = H^2 - w^{-4}$ reveals that

$$e_2((e_2(w))^2 + H^2w^2 + w^{-2}) = 0.$$

Using this in combination with fact that $E_1 w = 0$ and Lemma 5.1, there is a constant C such that

$$(e_2(w))^2 + H^2w^2 + w^{-2} + 2H = C. \tag{5.3}$$

Estimating (5.3) using the inequality $H^2w^2 + w^{-2} \geq 2H$ reveals that $C \geq 4H$, and the identity in (iii) follows from (5.3) using that $(wH + w^{-1})^2 = w^2\lambda_1^2$.

Item (iv) follows from a calculation using Corollary 4.7 and item (iii). $\square$

Remark 5.4.

- The inequality in 5.2(iii) is an equality precisely when $\mu = H$, in which case M is a cylinder with radius $1/2H$.
- It is possible to solve for w explicitly using the identity in 5.2(iii); see Remark 5.11 for more details.

We now consider the orthonormal frame $\{v_1, v_2, v_3\}$ defined on M by

$$v_1 = e_1, \quad v_2 = \frac{\overline{\nabla}_{e_1} e_1}{|\overline{\nabla}_{e_1} e_1|}, \quad v_3 = v_1 \times v_2.$$

Lemma 5.5. *Fix $p \in M$. We have*

$$\begin{aligned} v_1 &= \cos(\sqrt{C}s)\mathbf{a} + \sin(\sqrt{C}s)\mathbf{b}, \\ v_2 &= -\sin(\sqrt{C}s)\mathbf{a} + \cos(\sqrt{C}s)\mathbf{b}, \\ v_3 &= \mathbf{c}, \end{aligned}$$

where $\mathbf{a} = v_1(p), \mathbf{b} = v_2(p)$, and $\mathbf{c} = v_3(p)$.

Proof. The conclusion will follow once we show that

$$\begin{array}{lll} E_1 v_1 = \sqrt{C} v_2, & E_1 v_2 = -\sqrt{C} v_1, & E_1 v_3 = 0, \\ E_2 v_1 = 0, & E_2 v_2 = 0, & E_2 v_3 = 0. \end{array}$$

By using the definitions and Lemma 5.2(iv), we compute

$$\begin{aligned} E_1 v_1 &= \frac{1}{\sqrt{\mu}} \overline{\nabla}_{e_1} e_1 = w \overline{\nabla}_{e_1} e_1 = \sqrt{C} v_2, \\ E_2 v_1 &= \overline{\nabla}_{e_2} e_1 = \nabla_{e_2} e_1 = 0. \end{aligned} \tag{5.6}$$

Next, using that $v_3 = \frac{w}{\sqrt{C}} \left(\lambda_1 e_2 + \frac{e_2(\mu)}{2\mu} \nu \right)$, Lemma 5.2(i), $e_1(w) = 0$, and the calculations of the covariant derivatives of $\lambda_1 e_2 + \frac{e_2(\mu)}{2\mu} \nu$ in Lemma 4.8, we compute that

$$E_1 v_3 = E_2 v_3 = 0. \tag{5.7}$$

Finally, using the identities

$$0 = E_\alpha \langle v_i, v_j \rangle = \langle E_\alpha v_i, v_j \rangle + \langle v_i, E_\alpha v_j \rangle,$$

which hold for any $\alpha \in \{1, 2\}$ and $i, j \in \{1, 2, 3\}$, we conclude from (5.6) and (5.7) that

$$E_1 v_2 = -\sqrt{C} v_1, \quad E_2 v_2 = 0.$$

This completes the proof. □

The proof of Theorem 1.2 follows from the following rotationally symmetric parametrization of M.

Proposition 5.8. *The map $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by*

$$X(s,t) = \frac{w(t)\sin(\sqrt{C}s)}{\sqrt{C}} \mathbf{a} - \frac{w(t)\cos(\sqrt{C}s)}{\sqrt{C}} \mathbf{b} + z(t)\mathbf{c} \tag{5.9}$$

where

$$z(t) = \frac{1}{\sqrt{C}} \int_0^t \lambda_1(u) w(u) du.$$

is a parametrization of M.

Proof. By using Corollary 4.7 and Lemma 5.2, we compute that

$$v_2 = -\frac{e_2(w)}{\sqrt{C}} e_2 - \frac{\lambda_1 w}{\sqrt{C}}\nu, \quad v_3 = \frac{\lambda_1 w}{\sqrt{C}} e_2 - \frac{e_2(w)}{\sqrt{C}}\nu. \tag{5.10}$$

We then express the frame $\{e_1, e_2, \nu\}$ in terms of $\{v_1, v_2, v_3\}$: we have $e_1 = v_1$ and by (5.10) that

$$e_2 = -\frac{e_2(w)}{\sqrt{C}} v_2 + \frac{\lambda_1 w}{\sqrt{C}} v_3, \quad \nu = -\frac{\lambda_1 w}{\sqrt{C}} v_2 - \frac{e_2(w)}{\sqrt{C}} v_3.$$

In combination with Lemma 5.5, we find that

$$\begin{aligned} e_1 &= \cos(\sqrt{C}s)\mathbf{a} + \sin(\sqrt{C}s)\mathbf{b}, \\ e_2 &= \frac{e_2(w)}{\sqrt{C}} \sin(\sqrt{C}s)\mathbf{a} - \frac{e_2(w)}{\sqrt{C}} \cos(\sqrt{C}s)\mathbf{b} + \frac{\lambda_1 w}{\sqrt{C}}\mathbf{c}, \\ \nu &= \frac{\lambda_1 w}{\sqrt{C}} \sin(\sqrt{C}s)\mathbf{a} - \frac{\lambda_1 w}{\sqrt{C}} \cos(\sqrt{C}s)\mathbf{b} - \frac{e_2(w)}{\sqrt{C}}\mathbf{c}. \end{aligned}$$

From these identities, it is easily verified using (5.9) that $\frac{\partial X}{\partial s} = E_1 = \frac{e_1}{\sqrt{\mu}}$ and $\frac{\partial X}{\partial t} = E_2 = e_2$. Therefore X is a parametrization of M, and moreover M has an axis of revolution parallel to $\mathbf{c}$. □

Remark 5.11. It is possible to use Lemma 5.2(iii) to derive an explicit formula for w. To see this, recall from (5.3) that

$$(e_2(w))^2 + w^{-2} + H^2 w^2 + 2H = C.$$

Multiplying by $4w^2$ and setting $y = w^2$, we find

$$(e_2(y))^2 + 4H^2 \left(y^2 - y\frac{C - 2H}{4H^2} \right) + 4 = 0.$$

By completing the square, we find that

$$(e_2(y))^2 + 4H^2 \left(y - \frac{C - 2H}{2H^2} \right)^2 - \frac{C^2 - 4CH}{H^2} = 0. \tag{5.12}$$

The unique solutions of (5.12) are $y = \frac{C-2H}{2H^2} + \frac{\sqrt{C^2-4CH}}{2H^2} \sin(2Hu + \theta_0)$, where $\theta_0 \in \mathbb{R}$. Since $w \geq 0$, we find that

$$w(u) = \sqrt{\frac{C - 2H}{2H^2} + \frac{\sqrt{C^2 - 4CH}}{2H^2} \sin(2Hu + \theta_0)}$$

for some constant θ_0. Hence w is periodic with period $\frac{\pi}{H}$.

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