A Note Regarding Hopf's Umlaufsatz

Peter McGrath

Abstract. We note an argument proving simultaneously Hopf's rotation angle theorem and the C^1 Jordan curve theorem.

The Jordan curve theorem and Hopf rotation angle theorem are fundamental results about simple, closed, plane curves (hereafter referred to as *Jordan curves*). The first of these asserts that a Jordan curve bounds exactly two regions: an interior and exterior. The second asserts that the net angle the tangent vector of a positively oriented C^1 Jordan curve rotates as the curve is traversed is 2π. The purpose of this note is to unite these theorems with a single proof using a weak tubular neighborhood theorem proved below. While the application to the Jordan curve theorem is standard [**4**, Sec. 2.1], it appears that the application of the tubular neighborhood theorem to the rotation angle theorem may be new.

Hopf's theorem (see [**1**] for an exposition by Hopf), sometimes called the Umlaufsatz, is often used [**2**, Chap. 9] in proofs of the Gauss–Bonnet theorem. The Jordan curve theorem for general continuous curves is very subtle (see, however, [**5**] for an accessible proof), and we restrict our considerations to C^1 curves.

Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and $S^1 \subset \mathbb{R}^2$ be the unit circle. Let $\Pi : \mathbb{R} \to S^1$ be the covering map defined by $\Pi(x) = (\cos 2\pi x, \sin 2\pi x)$. We may identify S^1 with $[0, 1]/*$, where $*$ is the equivalence relation generated by requiring that $0 * 1$. Using this identification, any function $f : S^1 \to \mathbb{R}^2$ may be regarded as a function $f : [0, 1] \to \mathbb{R}^2$, where $f(0) = f(1)$.

Given $p \in \mathbb{R}^2$ and $r > 0$, let $B_r(p) = \{q \in \mathbb{R}^2 : |q - p| \leq r\}$.

The *winding number* or *degree* $\deg(f)$ of a continuous curve $f : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is an integer (see [**3**] for an elementary but rigorous introduction) which intuitively corresponds to $\frac{1}{2\pi}$ times the net change in oriented angle f makes with a fixed reference direction as the curve is traversed. More precisely, given any such f, we define $\deg(f) := \tilde{f}(1) - \tilde{f}(0)$, where $\tilde{f} : S^1 \to \mathbb{R}$ is the lift of f uniquely determined by requiring that

$$\Pi \circ \tilde{f} = \frac{f}{|f|} \quad \text{and} \quad \tilde{f}(0) \in [0, 1). \tag{1}$$

The winding number is a homotopy invariant: if f and g are homotopic (written $f \sim g$), then $\deg(f) = \deg(g)$.

A closed curve $\gamma : S^1 \to \mathbb{R}^2$ is C^1 if its component functions $x(t)$ and $y(t)$ are continuously differentiable and *regular* if $\dot{\gamma}(t) \neq 0$ for all $t \in [0, 1]$. The *rotation index* of a regular curve $\gamma : S^1 \to \mathbb{R}^2$ is defined to be $\deg(\dot{\gamma})$. For such a curve γ, we define the

doi.org/10.1080/00029890.2018.1449540
MSC: Primary 55M25

reversal $\gamma^r : S^1 \to \mathbb{R}^2$ by $\gamma^r(t) = \gamma(1-t)$. γ^r is the same curve as γ, but traversed in the opposite direction, and the rotation index of γ^r is minus the rotation index of γ. If $\gamma : S^1 \to \mathbb{R}^2$ is a closed plane curve and $p \in \mathbb{R}^2 \setminus \gamma$ (where here, as elsewhere in this note, we abuse notation slightly by identifying γ with its image), the *winding number* of γ about p is defined to be $\deg(\gamma - p)$.

Theorem 1. *Let γ be a regular, C^1 Jordan curve. After possibly replacing γ by γ^r:*

1. *(Umlaufsatz) The rotation index of γ is one.*
2. *(Jordan curve theorem) $\mathbb{R}^2 \setminus \gamma$ is a disjoint union of two path components, the interior, $\operatorname{int}(\gamma)$, and exterior, $\operatorname{ext}(\gamma)$, which are characterized by*

$$\deg(\gamma - p) = \begin{cases} 1 & \textit{if} \quad p \in \operatorname{int}(\gamma) \\ 0 & \textit{if} \quad p \in \operatorname{ext}(\gamma). \end{cases}$$

Proof. By compactness, the image of γ contains a point with smallest y-coordinate; suppose without loss of generality that $\gamma(0)$ is such a point. After possibly replacing γ by its reversal γ^r, we may suppose $\dot\gamma(0) = ce_1$ for some $c > 0$. Choose the continuous unit normal field ν along γ such that $\nu_{\gamma(0)} = -e_2$.

For $\epsilon \in \mathbb{R}$, consider the curve γ_ϵ defined by $\gamma_\epsilon(t) := \gamma(t) - \epsilon\nu_{\gamma(t)}$. When $|\epsilon'|$ is small, $\gamma_{\epsilon'}$ should be thought of a "parallel curve" to γ. The tubular neighborhood theorem [**2**, Theorem 10.19] asserts that for a smooth Jordan curve γ and $\epsilon > 0$ small enough, the map from $\gamma \times (-\epsilon, \epsilon)$ to $\mathbb{R}^2$ defined by $(p, \epsilon') \mapsto \gamma_{\epsilon'}(p)$ is a diffeomorphism. For our purposes, the following less general version—proved at the end of the note—will suffice.

Assertion. There exists $\epsilon > 0$ such that:

(i). For all ϵ' such that $0 < |\epsilon'| \le \epsilon$, the curves γ and $\gamma_{\epsilon'}$ have disjoint images.
(ii). For all $t \in [0, 1]$, if $\gamma(t) \in B_\epsilon(\gamma(0))$, then the y-component of $\nu_{\gamma(t)}$ is negative.

Now fix some $\epsilon > 0$ such that the assertion holds and define $p_{\text{in}} := \gamma_\epsilon(0)$, $p_{\text{out}} := \gamma_{-\epsilon}(0)$. We claim that

(a) $\deg(\gamma - p_{\text{in}}) = 1$ and $\deg(\gamma - p_{\text{out}}) = 0$.
(b) $\gamma - p_{\text{in}} \sim \dot\gamma$.

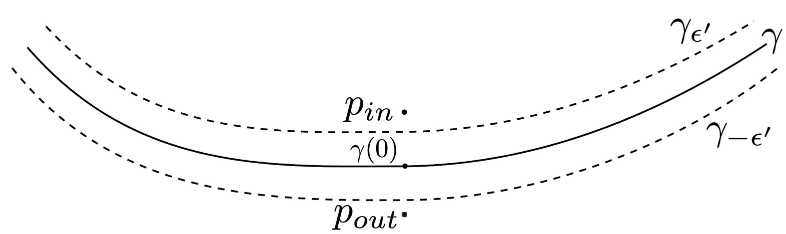

Figure 1. Portions of the curves γ, $\gamma_{\epsilon'}$, and $\gamma_{-\epsilon'}$, where $\epsilon' \in (0, \epsilon]$.

Intuitively (a) encodes the fact that γ winds once around p_{in} and zero times around p_{out}, since p_{in} lies just above the lowest point of γ, while p_{out} is entirely below γ (see Figure 1).

The Umlaufsatz follows immediately from combining the first part of (a) with (b). For the Jordan curve theorem, note that since $\deg(\gamma - p)$ is a locally constant function of $p \in \mathbb{R}^2 \setminus \gamma$, (a) implies that $\mathbb{R}^2 \setminus \gamma$ has at least two path components. We will show that each point in $\mathbb{R}^2 \setminus \gamma$ can be connected by a path in $\mathbb{R}^2 \setminus \gamma$ to a point on either γ_ϵ or $\gamma_{-\epsilon}$, and it will follow from this that $\mathbb{R}^2 \setminus \gamma$ has at most two path components.

Fix $p \in \mathbb{R}^2 \setminus \gamma$ and let q be a point on γ that is closest to p. By elementary geometry, the line segment connecting p and q is orthogonal to the tangent line to γ at q; hence $p = q + \delta \nu_q$ for some $\delta \neq 0$. If $\delta > 0$, consider the point $r := q + \epsilon \nu_q$, which is on $\gamma_{-\epsilon}$. Each point on the line segment from p to r is either closer to p than q is and consequently not on γ by the choice of q, or on the curve $\gamma_{-\epsilon'}$ for some $0 < \epsilon' \leq \epsilon$ and therefore not on γ by (i). Therefore, the line segment from p to r lies in $\mathbb{R}^2 \setminus \gamma$. If $\delta < 0$, we take $r := q - \epsilon \nu_q$, which is on γ_ϵ and argue in the same way. We conclude that $\mathbb{R}^2 \setminus \gamma$ has at most two path components, and this completes the proof of the Jordan curve theorem.

It remains to prove (a) and (b). We first claim that

$$(\gamma - p_{\text{in}})^{-1} \left(\{(0, y) \in \mathbb{R}^2 : y < 0\}\right) = \{0, 1\}. \tag{2}$$

By the choice of $\gamma(0)$, if $\gamma(t) - p_{\text{in}}$ were on the negative y-axis for some $t \notin \{0, 1\}$, then $\gamma(t)$ would lie on $\gamma_{\epsilon'}$ for some ϵ' satisfying $0 < \epsilon' < \epsilon$, which would contradict (i). Let Γ be the lift of $\gamma - p_{\text{in}}$ as in (1). Since $\gamma - p_{\text{in}}$ is C^1 and $\dot\gamma(0) = e_1$, it follows from (2) and the intermediate value theorem that $\deg(\gamma - p_{\text{in}}) = \Gamma(1) - \Gamma(0) = 1$. By the choice of $\gamma(0)$, the image of $\gamma - p_{\text{out}}$ omits the entire negative y-axis, so an argument similar to the one above shows $\deg(\gamma - p_{\text{out}}) = 0$. This proves (a).

For (b), define $H : [0, 1]^2 \to \mathbb{R}^2 \setminus \{0\}$ by

$$H(s, t) = \gamma(t) - \gamma_\epsilon(s).$$

Notice that $\epsilon \nu_\gamma$ is equal to the diagonal of H (i.e., $\epsilon \nu_{\gamma(t)} = H(t, t)$). The diagonal of H is also homotopic to the concatenation V of the left and top sides of the domain square given by

$$V(t) = \begin{cases} H(0, 2t) & \text{if} \quad t \in [0, 1/2] \\ H(2t - 1, 0) & \text{if} \quad t \in [1/2, 1]. \end{cases}$$

We now claim that the image of the second loop in this product omits the entire positive y-axis and hence is nullhomotopic. To see this, suppose for the sake of a contradiction that there exists $t \in S^1$ such that $\gamma(0) - \gamma_\epsilon(t)$ lies on the positive y-axis. The choice of $\gamma(0)$ implies that $\gamma(t) - \gamma(0)$ and $\gamma(t) - \gamma_\epsilon(t) = \epsilon \nu_{\gamma(t)}$ each have positive y-components and that $|\gamma(0) - \gamma(t)| < \epsilon$. This contradicts (ii).

Consequently, since $\gamma - p_{\text{in}} = H(0, \cdot)$, it follows that $\gamma - p_{\text{in}} \sim \nu_\gamma$. Since the maps ν_γ and $\dot\gamma$ differ by constant angle $\frac{\pi}{2}$, we see that $\nu_\gamma \sim \dot\gamma$; hence $\gamma - p_{\text{in}} \sim \dot\gamma$ and (b) holds. ■

Proof of Assertion. We first show there exists $\epsilon > 0$ such that (i) holds. Suppose for the sake of a contradiction that for each $k \in \mathbb{N}$, there exists ϵ_k such that $0 < |\epsilon_k| \leq \frac{1}{k}$ and $t_k, t_k' \in [0, 1]$ such that $\gamma(t_k) = \gamma_{\epsilon_k}(t_k')$. By compactness, we may suppose that both $(t_k)_{k \in \mathbb{N}}$ and $(t_k')_{k \in \mathbb{N}}$ converge. Since γ is embedded, both sequences have the same limit, say t_∞. By a change of coordinates, we may suppose $\gamma(t_\infty) = \mathbf{0}$ and $\dot\gamma(t_\infty) = ce_1$, for some $c > 0$. Since γ is C^1, there exists $\delta > 0$ such that, for all t sufficiently close to t_∞, $\gamma(t)$ lies on the graph of a function $f \in C^1((-\delta, \delta))$ satisfying $f(0) = f'(0) = 0$. Then for all sufficiently large $k \in \mathbb{N}$, there exist $x_k, x_k' \in (-\delta, \delta)$ satisfying $\lim_{k \to \infty} x_k = \lim_{k \to \infty} x_k' = 0$ such that

$$\gamma(t_k) = (x_k, f(x_k)), \quad \gamma_{\epsilon_k}(t_k') = (x_k', f(x_k')) - \frac{\epsilon_k (f'(x_k'), -1)}{\sqrt{1 + \left(f'(x_k')\right)^2}}. \tag{3}$$

From the assumption that $\gamma(t_k) = \gamma_{\epsilon_k}(t'_k)$ and (3), we conclude

$$\frac{f(x_k) - f(x'_k)}{x_k - x'_k} = -\frac{1}{f'(x'_k)}, \tag{4}$$

which is clearly a contradiction, since (because f is C^1) the left side of (4) approaches zero as $k \to \infty$, while the magnitude of the right-hand side is unbounded as $k \to \infty$.

We next show that (ii) holds as long as $\epsilon > 0$ is small enough. Since $\nu_{\gamma(0)} = -e_2$, choose (by continuity) $\delta > 0$ such that if $0 \le t < \delta$ or $1 - \delta < t \le 1$ then $\nu_{\gamma(t)}$ has negative y-component. If $\epsilon > 0$ is sufficiently small, then $B_\epsilon(\gamma(0))$ is disjoint from the image of $[\delta, 1-\delta]$ under γ (since this image is compact). For such a choice of ϵ, if $\gamma(t) \in B_\epsilon(\gamma(0))$, then either $t < \delta$ or $t > 1 - \delta$, and therefore $\nu_{\gamma(t)}$ has negative y-component. ■

ACKNOWLEDGMENT. The author thanks the referee for many helpful comments and suggestions that improved the paper.

REFERENCES

[1] Hopf, H. (1983). *Differential Geometry in the Large*. Lecture Notes in Mathematics, Vol. 1000. (Notes taken by Peter Lax and John Gray, With a preface by S. S. Chern.) Berlin: Springer-Verlag.

[2] Lee, J. (1997). *Riemannian Manifolds*. Graduate Texts in Mathematics, Vol. 176. New York, NY: Springer-Verlag.

[3] Munkres, J. (1975). *Topology: A First Course*. Englewood Cliffs, NJ: Prentice-Hall, Inc.

[4] Tapp, K. (2016). *Differential Geometry of Curves and Surfaces*. Undergraduate Texts in Mathematics. Switzerland: Springer International Publishing.

[5] Thomassen, C. (1992). The Jordan–Schönflies theorem and the classification of surfaces. *Amer. Math. Monthly*. 99(2): 116–131.

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104
pjmcgrat@sas.upenn.edu